Series expansions of three flavour neutrino oscillation formulas

Robert Johansson
Neutrinos

- Particles with very small mass (< 1 eV)
- Very small cross-sections => Difficult to detect
- Three flavours (types): $\nu_e$, $\nu_\mu$, $\nu_\tau$
- Fermions with no electric charge
Neutrino oscillations

- Means that the neutrinos can change flavour.
- Evidence for neutrino oscillations from experiments, Super-Kamiokande and SNO.
- Occur if the neutrino masses are different and if the flavour and mass states are different.
- Indicate physics beyond the Standard Model.
Two flavour neutrino oscillations in vacuum

- Neutrino oscillation probability
  \[ P_{\text{change}} = \sin^2 2\theta \sin^2 \frac{\Delta m^2 L}{4E} \]

- One mixing angle \( \theta \). The mixing matrix is
  \[ U = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \]

- One mass-squared difference
  \[ \Delta m^2 \equiv m_2^2 - m_1^2 \]
Matter effects

- Additional term in the Hamiltonian because of interactions between electron neutrinos and electrons.

- The neutrino oscillation probability change to

\[ P_{\text{change}} = \frac{\sin^2 2\theta}{C^2} \sin^2 \frac{\Delta m^2 L C}{4E} \]

\[ C = \sqrt{\sin^2 2\theta + \left( \cos 2\theta - \frac{2E}{\Delta m^2 V_{CC}} \right)^2} \]

- Matter effects are only clearly visible for larger values of L/E
Three flavours

- The mixing matrix becomes a 3x3-matrix

\[
U = \begin{pmatrix}
    c_{13}c_{12} & s_{12}c_{13} & s_{13}e^{-i\delta} \\
    -s_{12}c_{23} - s_{23}s_{13}c_{12}e^{i\delta} & c_{23}c_{12} - s_{23}s_{13}s_{12}e^{i\delta} & s_{23}c_{13} \\
    s_{23}s_{12} - s_{13}c_{23}c_{12}e^{i\delta} & -s_{23}c_{12} - s_{13}s_{12}c_{23}e^{i\delta} & c_{23}c_{13}
\end{pmatrix}
\]

- Two mass squared differences, $\Delta m^2_{31}$ and $\Delta m^2_{21}$.

- Leads to very complicated formulas
Series expansions

• Simplify the complicated analytical expressions

• Expansion parameters, $\theta_{13}$ and $\alpha$. $\alpha = 0.026$.

\[ \alpha \equiv \frac{\Delta m_{21}^2}{\Delta m_{31}^2} \]

• Three different expansions
  - Expansion in $\alpha$ up to first order.
  - Expansion in $\theta_{13}$ up to first order.
  - Expansion in $\alpha$ and $\theta_{13}$ up to second order.
Important features

• Simplicity

• Accuracy
Series expansions in $\theta_{13}$ and $\alpha$.

\[ P_{e \rightarrow \nu_e} = 1 - \alpha^2 \sin^2 2\theta_{12} \frac{\sin^2 A\Delta}{A^2} - 4 s_{13}^2 \frac{\sin^2 (A - 1)\Delta}{(A - 1)^2} \]

\[ P_{e \rightarrow \nu_{\mu}} = \alpha^2 \sin^2 2\theta_{12} c_{23}^2 \frac{\sin^2 A\Delta}{A^2} + 4 s_{13}^2 s_{23}^2 \frac{\sin^2 (A - 1)\Delta}{(A - 1)^2} \]

\[ + 2\alpha \sin 2\theta_{12} s_{13} \sin 2\theta_{23} \cos(\Delta - \delta) \frac{\sin A\Delta}{A} \frac{\sin (A - 1)\Delta}{A - 1} \]

\[ A \equiv \frac{2E V_{CC}}{\Delta m_{31}^2} \quad \Delta \equiv \frac{\Delta m_{31}^2 L}{4E} \]
Conclusions

- The series expansions in $\theta_{13}$ and $\alpha$ up to second order are the most useful expansion because:
  
  The simplest and shortest expressions

  Accuracy does not depend so much on the value of $\theta_{13}$

  Accurate for a large range of $L$ and $E$ values