Duality in the seesaw mechanism of neutrino mass generation

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Neutrinos are massive!

Data from the atmospheric, solar, reactor and accelerator neutrino experiments: Neutrinos oscillate \( m_\nu \neq 0 \).

First direct evidence for physics beyond the standard model of particle physics!

All the data (except LSND) is described by oscillations between 3 weak (flavour) eigenstate neutrinos \((\nu_e, \nu_\mu, \nu_\tau)\) which are linear combinations of 3 massive neutrinos \((\nu_1, \nu_2, \nu_3)\). From the oscillation experiments:

\[
\begin{align*}
\Delta m^2_{21} & \equiv m_2^2 - m_1^2 \simeq 8 \cdot 10^{-5} \text{ eV}^2 \\
\Delta m^2_{31} & \equiv m_3^2 - m_1^2 \simeq 2.5 \cdot 10^{-3} \text{ eV}^2 
\end{align*}
\]

From cosmology and \(\beta\)-decay experiments: \( m_\nu \lesssim 1 \text{ eV} \)

If neutrino masses are hierarchical (like masses of the other fermions)

\[
m(\nu_1) \ll m(\nu_2) \approx 10^{-2} \text{ eV}, \quad m(\nu_3) \approx 5 \cdot 10^{-2} \text{ eV}
\]
If neutrinos are almost degenerate in mass with only $\Delta m^2$ hierarchical, in any case $m_\nu < 1$ eV.

$\Rightarrow$ More than 5 (possibly more than 7) orders of magnitude smaller than the electron mass!

Can we understand that?

◊ An attractive solution: The seesaw mechanism
Fermion masses

The fermion mass Lagrangian:

\[ \mathcal{L}_m \ni m \bar{\Psi} \Psi = m (\bar{\Psi}_L \Psi_R + \bar{\Psi}_R \Psi_L), \]

\[ \Psi_{R,L} = \frac{1 \pm \gamma_5}{2} \Psi, \quad \Psi = \Psi_L + \Psi_R, \]

*Fields* \( L \) and \( R \) fields are necessary to make up a fermion mass

Dirac fermions: \( \Psi_L \) and \( \Psi_R \) are completely independent fields

Majorana fermions: \( \Psi_R = (\Psi_L)^c \), where \( (\Psi)^c \equiv C \bar{\Psi}^T \) is charge-conjugate field \( (C = i\gamma_2\gamma_0) \quad \Rightarrow \)

\[ \mathcal{L}^{Maj}_m \ni \frac{m}{2} [\bar{\Psi}_L (\Psi_L)^c + h.c.] \sim \Psi_L \Psi_L + (\Psi_L)^c (\Psi_L)^c \]

Break all charges (electric, lepton, baryon, color) – can only be written for entirely neutral fermions \( \Rightarrow \) Neutrinos are the only known candidates
Fermion masses in the Standard Model

In the SM (gauge group $SU(2)_L \times U(1)$): all left-handed fermions are in doublets of $SU(2)_L$, all right-handed fermions are $SU(2)_L$ - singlets $\Rightarrow$
direct mass terms $\bar{\Psi}_L \Psi_R$ forbidden (violate gauge symmetry – break the renormalizability). Higgs mechanism: Yukawa couplings with the doublet of scalar Higgs fields:

$$\mathcal{L} \supset Y \bar{L} R H + h.c., \quad H = \begin{pmatrix} H^+ \\ H^0 \end{pmatrix}$$

After $H^0$ gets a VEV ($\langle H^0 \rangle = v \approx 174$ GeV), the mass term is generated:

$$\mathcal{L} \supset m \bar{L} R + h.c., \quad m = Y \langle H^0 \rangle = Yv$$
Why are neutrinos so light?

In the minimal SM: \( m_\nu = 0 \). Add 3 RH \( \nu \)'s \( N_{Ri} \):

\[
- \mathcal{L}_Y \supset Y_\nu \bar{L}_L N_R H + h.c., \quad l_{Li} = \begin{pmatrix} \nu_{Li} \\ e_{Li} \end{pmatrix}
\]

\( \langle H^0 \rangle = v = 174 \text{ GeV} \) \( \Rightarrow \) \( m_\nu = m_D = Y_\nu v \)

\( m_\nu < 1 \text{ eV} \) \( \Rightarrow \) \( Y_\nu < 10^{-11} \) – Not natural!

Is it a problem? \( Y_e \simeq 3 \times 10^{-6} \). But: with \( m_\nu \neq 0 \), huge disparity between the masses within each fermion generation!

A simple and elegant mechanism – **seesaw**

(Minkowski, 1977; Gell-Mann, Ramond & Slansky, 1979; Yanagida, 1979; Glashow, 1979; Mohapatra & Senjanović, 1980)
Heavy $N_{Ri}$'s make $\nu_{Li}$'s light:

\[-\mathcal{L}_{Y+m} = Y_\nu \bar{L}_L N_R H + \frac{1}{2} M_R N_R N_R + h.c.,\]

Neutrino mass matrix in the $(\nu_L, (N_R)^c)$ basis:

\[M_\nu = \begin{pmatrix} 0 & m_D \\ m_D^T & M_R \end{pmatrix}\]

$N_{Ri}$ are EW singlets $\Rightarrow M_R \sim M_{GUT}(M_I) \gg m_D \sim \nu$.

Block diagonalization: $M_N \simeq M_R$,

\[m_{\nu_L} \simeq -m_D M_R^{-1} m_D^T \Rightarrow m_\nu \sim \frac{(174 \text{ GeV})^2}{M_R}\]

For $m_\nu \lesssim 0.05 \text{ eV} \Rightarrow M_R \gtrsim 10^{15} \text{ GeV} \sim M_{GUT} \sim 10^{16} \text{ GeV}!$
Baryogenesis via Leptogenesis

(Fukugita & Yanagida, 1986; Luty, 1992; Covi et al., 1996; Buchmüller & Plümacher, 1996; ...)

◊ Seesaw has a built-in mechanism for generating the baryon asymmetry of the Universe! Observations:

\[
\eta_B = \frac{n_B - n_{\bar{B}}}{n_{\gamma}} = (6.3 \pm 0.3) \times 10^{-10}
\]

Three Sakharov’s conditions for generating an asymmetric Universe starting from a \( \Delta B = 0 \) state:

- Baryon number non-conservation
- C and CP violation
- Deviation from thermal equilibrium

Baryogenesis via leptogenesis satisfies all of them!
Baryogenesis via leptogenesis – contd.

(1) Out-of-equilibrium CP and L violating decay of $N_1 \Rightarrow$ a net $L \neq 0$ is produced

$L$ violation (due to Majorana nature of $N_i$): $N_i \rightarrow lH$, $N_i \rightarrow \bar{l}\bar{H}$

CP violation: $\Gamma(N_i \rightarrow lH) \neq \Gamma(N_i \rightarrow \bar{l}\bar{H})$

Out-of-equilibrium decay condition: $\Gamma_1 < H(T = M_1)$

(2) Reprocessing of the produced $L$ into $B$ by electroweak sphalerons

SM: At tree level, $B$ and $L$ are conserved. Broken at 1-loop level by triangle anomalies. But: $\Delta B = \Delta L \Rightarrow B - L$ is conserved!

Nonperturbative EW field configurations – sphalerons: conserve $B - L$ but efficiently wash out $B + L$ for

$$10^2 \text{ GeV} \lesssim T \lesssim 10^{12} \text{ GeV}$$

◊ Viable $\eta_B$ produced for $M_1 \gtrsim 10^9$ GeV (for non-degenerate $N_i$'s)
Type II seesaw

Type I seesaw: Neutrino mass generated through exchange of a heavy RH \( \nu \)'s \( N_R \); in type II – through exchange of a heavy \( SU(2)_L \)-triplet scalars \( \Delta_L \):

Why Higgs triplets?

In the SM, RH neutrinos \( N_R \) are singlets of the gauge group – “aliens”. They are more natural in Left-Right symmetric extensions of the SM: \( SU(2)_L \times SU(2)_R \times U(1)_{B-L} \), \( SU(2)_L \times SU(2)_R \times SU(4)_{PS} \), \( SO(10) \), ...

LR symmetric models explain in a nice way maximal P- and C-violation in low-\( E \) weak interactions as a spontaneous symmetry breaking phenomenon.
In LR - symmetric models:

$N_R$ are not singlets – they are parts of $SU(2)_R$ doublets, along with $e_R$:

$$l_{Li} = \begin{pmatrix} \nu_{Li} \\ e_{Li} \end{pmatrix}, \quad l_{Ri} = \begin{pmatrix} N_{Ri} \\ e_{Ri} \end{pmatrix}$$

No direct mass of $N_R$ can be introduced $\Rightarrow M_R$ is obtained through the Higgs mechanism due to Yukawa interaction of $N_R$ with the $SU(2)_R$ - doublet Higgs $\Delta_R$:

$$-\mathcal{L}_Y \ni f^*_R N_R N_R \Delta^0_R + h.c.; \quad \langle \Delta^0_R \rangle = v_R \Rightarrow M_R = f^*_R v_R$$

LR symmetry: if $\Delta_R$ exists, there must also be an $SU(2)_L$ - triplet Higgs $\Delta_L$:

$$-\mathcal{L}_Y \ni f_L \nu_L \nu_L \Delta^0_L + h.c.; \quad \langle \Delta^0_L \rangle = v_L \Rightarrow m_L = f_L v_L$$

Direct mass term for $\nu_L$ is obtained. Experimental upper limit on its VEV: $v_L \lesssim 1$ MeV (from the $\rho$ parameter – the ratio of $M_W$ and $M_Z$).
In LR - symmetric models:

In general, both type I and type II seesaw contributions to $m_\nu$ are present:

$$
\mathcal{M}_\nu = \begin{pmatrix} m_L & m_D \\ m_D^T & M_R \end{pmatrix}
$$

$m_D$ comes from the Yukawa interactions with the bi-doublet Higgs $\Phi$:

$$
Y_1 \bar{l}_L \Phi l_R + Y_2 \bar{l}_L \Phi l_R, \quad \Phi = \tau_2 \Phi^* \tau_2 \quad \Rightarrow \quad m_D = y v
$$

Block diagonalization of $\mathcal{M}_\nu$:

$$
\therefore \quad m_\nu \simeq m_L - m_D M_R^{-1} m_D^T
$$

or

$$
\quad m_\nu \simeq f_L v_L - v^2 y (f_R v_R)^{-1} y^T
$$

In general: $m_\nu$, $f_L$ and $f_R$ – complex symmetric, $y$ – complex matrix.
Parity symmetry

Discrete LR symmetry (P) in LR-symm. models – two possible implementations:

(1) \( l_L \leftrightarrow l_R, \quad \Delta_L \leftrightarrow \Delta_R^*, \quad \Phi \leftrightarrow \Phi^\dagger \)

Yields

\[ f_L = f_R^*, \quad y = y^\dagger, \]

(2) \( l_L \leftrightarrow l_L^c \equiv (l_R^c)^c, \quad \Delta_L \leftrightarrow \Delta_R, \quad \Phi \leftrightarrow \Phi^T \)

Yields

\[ f_L = f_R \equiv f, \quad y = y^T. \]

Both implementations possible. Implem. (2) is more natural in \( SO(10) \) GUTs (is an automatic gauge symmetry)
Adhere to the second implementation

\[ m_\nu = f v_L - v^2 y (v_R f)^{-1} y \]

For realization (1) of parity: type I term contains \((f^*)^{-1}\) instead of \(f^{-1}\).

\(N_R, \Delta_R, \Delta_L\) are all at very high scale \((\sim 10^{12} - 10^{16} \text{ GeV})\) \Rightarrow no direct way of probing this sector of the theory.

◊ Neutrinos may provide a low-energy window into new physics at very high energy scales!

**Bottom-up approach:**

- Take \(m_\nu\) from experiment
- Take \(y\) from data + theoretical assumptions (quark-lepton symm., GUTs)
- Solve the seesaw relation for \(f\)

Mass matrix of light neutrinos \( m_\nu \)

Symmetric mass matrix – diagonalized as

\[
m_\nu = U^T m_{\text{diag}} U, \quad m_{\text{diag}} = \text{diag}(m_1, m_2, m_3)
\]

The masses \( m_i \) known up to the overall scale parameterized by \( m_1 \):

\[
m_2 = \sqrt{m_1^2 + \Delta m_{21}^2}, \quad m_3 = \sqrt{m_1^2 + \Delta m_{31}^2}
\]

Leptonic mixing matrix \( U \):

\[
U = \begin{pmatrix}
1 & 0 & 0 \\
0 & c_{23} & s_{23} \\
0 & -s_{23} & c_{23}
\end{pmatrix}
\begin{pmatrix}
c_{13} & 0 & s_{13}e^{-i\delta} \\
0 & 1 & 0 \\
-s_{13}e^{i\delta} & 0 & c_{13}
\end{pmatrix}
\begin{pmatrix}
c_{12} & s_{12} & 0 \\
-s_{12} & c_{12} & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & e^{i\alpha} & 0 \\
0 & 0 & e^{i\beta}
\end{pmatrix}
\]

\( c_{ij} \equiv \cos \theta_{ij}, \ s_{ij} \equiv \sin \theta_{ij} \)

Experiment: \( \theta_{12} \simeq 34^\circ, \ \theta_{23} \simeq 45^\circ, \ \theta_{13} < 10^\circ \).

CP violating phases can be varied between 0 and 2\( \pi \).
Inverting the seesaw formula

Pure type I:

\[ m_\nu = - \left( \frac{v^2}{v_R} \right) y f^{-1} y^T \Rightarrow f = - \left( \frac{v^2}{v_R} \right) y^T m_\nu^{-1} y \]

Pure type II:

\[ m_\nu = f v_L \Rightarrow f = m_\nu / v_L \]

General type I + II seesaw: a nonlinear matrix equation for \( f \). Yet can be readily solved analytically!

Important point: **A duality property of LR-symmetric seesaw**

\[ \frac{m_\nu}{v_L} - f = - \frac{v^2}{v_L v_R} y f^{-1} y \]

◊ If \( f \) is a solution, so is \( \hat{f} \equiv (m_\nu / v_L - f) \)

\[ \Rightarrow \text{There is always an even number of solutions! } (\#sol. = 2^n) \]
LR symmetry important for duality!

It was important that \( f_L = f_R \) and \( y^T = y \).

For the other realization of parity symmetry \( (f_L = f_R^* \text{ and } y = y^\dagger) \) exactly the same duality holds!

Two-generation case

\[
 f^{-1} = \frac{F}{f_{22}} \begin{pmatrix} f_{22} & -f_{12} \\ -f_{12} & f_{11} \end{pmatrix}, \quad y = \begin{pmatrix} y_{e1} & y_{e2} \\ y_{\mu1} & y_{\mu2} \end{pmatrix},
\]

\[ F \equiv \det f, \quad y_{\mu1} = y_{e2}. \] Define: \( x \equiv v_L v_{R}/v^2 \) and \( m \equiv m_\nu/v_L \Rightarrow \)

\[
x F (f_{11} - m_{ee}) = f_{22} y_{e1}^2 - 2 f_{12} y_{e1} y_{e2} + f_{11} y_{e2}^2,
\]

\[
x F (f_{22} - m_{\mu\mu}) = f_{22} y_{e2}^2 - 2 f_{12} y_{e2} y_{\mu2} + f_{11} y_{\mu2}^2,
\]

\[
x F (f_{12} - m_{e\mu}) = f_{22} y_{e1} y_{e2} - f_{12} (y_{e1} y_{\mu2} + y_{e2}^2) + f_{11} y_{e2} y_{\mu2},
\]
A system of coupled 3rd order equations

Admits a simple exact analytic solution!

Go to the basis where \( y \) is diagonal: \( y = \text{diag}(y_1, y_2) \) (no loss of generality)

\[
XF(f_{11} - m_{ee}) = f_{22} y_1^2 \\
XF(f_{22} - m_{\mu\mu}) = f_{11} y_2^2 \\
XF(f_{12} - m_{e\mu}) = -f_{12} y_1 y_2
\]

Rescaling:

\[
f_{ij} = \sqrt{\lambda} f'_{ij}, \quad m_{ij} = \sqrt{\lambda} m'_{ij}, \quad y_{ij} = \sqrt{\lambda} y'_{ij},
\]

\( \lambda \) - arbitrary; fix it by requiring \( F' \equiv \det f' = 1 \). The system of eqs. for \( f'_{ij} \) becomes linear

Express \( f'_{ij} \) back through unprimed \( m_{ij}, y_{1,2} \) and subst. into \( F' = 1 \) ⇒ 4th order characteristic equation for \( \lambda \)
The solution:

\[
f = \frac{x\lambda}{(x\lambda)^2 - y_1^2 y_2^2} \begin{pmatrix} x\lambda m_{ee} + y_1^2 m_{\mu\mu} & m_{e\mu}(x\lambda - y_1 y_2) \\ m_{e\mu}(x\lambda - y_1 y_2) & x\lambda m_{\mu\mu} + y_2^2 m_{ee} \end{pmatrix}
\]

\(\lambda\) has to be found from

\[
\left[(x\lambda)^2 - y_1^2 y_2^2\right]^2 - x \left[\det m(x\lambda - y_1 y_2)^2 x\lambda \\
+ (m_{ee} y_2 + m_{\mu\mu} y_1)^2 (x\lambda)^2\right] = 0
\]

Has 4 complex solutions \(\Rightarrow\) 4 solutions for \(f\) which form 2 dual pairs

Determinant of the seesaw relation \(x\hat{f} = -y f^{-1} y \Rightarrow x^2 F \hat{F} = y_1^2 y_2^2\)

\(F' = 1 \Rightarrow F = \lambda\); therefore

\[
x^2 \lambda \hat{\lambda} = y_1^2 y_2^2
\]

Allows to express the four solutions for \(\lambda\) in a simple closed form
The solutions

\[ x\lambda_i = \frac{1}{4}[x \det m + r_\pm \pm \sqrt{2(\det m)^2 x^2 + 4kx + 2xr_\pm \det m}], \]

where

\[ r_\pm = \pm \sqrt{(\det m)^2 x^2 + 4kx + 16y_1^2 y_2^2}, \]

\[ k = m_{ee}^2 y_2^2 + 2m_{e\mu} y_1 y_2 + m_{\mu\mu}^2 y_1^2. \]

When \( \lambda_1 \) satisfies \( |x\lambda_1| \gg |y_1 y_j| \Rightarrow f_1 \approx m \) (type II seesaw case)

The dual solution \( x\lambda_2 = x\hat{\lambda}_1 = y_1^2 y_2^2/(x\lambda_1) \) has modulus \( \ll |y_i y_j| \Rightarrow f_2 = \hat{f}_1 \) takes type I seesaw form
The solutions

\[ x \lambda_i = \frac{1}{4} \left[ x \det m + r_\pm \pm \sqrt{2(\det m)^2 x^2 + 4kx + 2xr_\pm \det m} \right], \]

where

\[ r_\pm = \pm \sqrt{(\det m)^2 x^2 + 4kx + 16y_1^2 y_2^2}, \]

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When \( \lambda_1 \) satisfies \( |x \lambda_1| \gg |y_i y_j| \Rightarrow f_1 \simeq m \) (type II seesaw case)

The dual solution \( x \lambda_2 = x \hat{\lambda}_1 = y_1^2 y_2^2 / (x \lambda_1) \) has modulus \( \ll |y_i y_j| \Rightarrow f_2 = \hat{f}_1 \) takes type I seesaw form

Condition \( |x \lambda_1| \gg |y_i y_j| \) can only be satisfied when \( |m_{\alpha\beta} m_{\gamma\delta}| \gg 4|y_i y_j / x|. \)

This ensures the existence of solutions with one seesaw type dominance.
The solutions

\[ x\lambda_i = \frac{1}{4}[x\det m + r_\pm \pm \sqrt{2(\det m)^2 x^2 + 4kx + 2xr_\pm \det m}], \]

where

\[ r_\pm = \pm \sqrt{(\det m)^2 x^2 + 4kx + 16y_1^2 y_2^2}, \]

\[ k = m_{ee}^2 y_2^2 + 2m_{e\mu} y_1 y_2 + m_{\mu\mu}^2 y_1^2. \]

When \( \lambda_1 \) satisfies \( |x\lambda_1| \gg |y_i y_j| \) \( \Rightarrow f_1 \approx m \) (type II seesaw case)

The dual solution \( x\lambda_2 = x\hat{\lambda}_1 = y_1^2 y_2^2 / (x\lambda_1) \) has modulus \( \ll |y_i y_j| \) \( \Rightarrow f_2 = \hat{f}_1 \) takes type I seesaw form

Condition \( |x\lambda_1| \gg |y_i y_j| \) can only be satisfied when \( |m_{\alpha\beta} m_{\gamma\delta}| \gg 4|y_i y_j / x|. \)

This ensures the existence of solutions with one seesaw type dominance.

But: in general not all 4 solutions correspond to one seesaw type dominance in this limit: if \( |\det m| \gg 4|y_i y_j / x| \), the remaining 2 dual solutions \( f_3 \) and \( f_4 \)

are of mixed type. When \( |m_{\alpha\beta} m_{\gamma\delta}| \lesssim 4|y_i y_j / x| \), all 4 sols. are of mixed type.
Three lepton generations

Once again go to the basis where $y = diag(y_1, y_2, y_3)$.

$$f^{-1} = \frac{1}{F} \begin{pmatrix} f_{22}f_{33} - f_{23}^2 & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}$$

System of 6 coupled 4th order equations:

$$xF(m_{11} - f_{11}) = y_1^2 (f_{22}f_{33} - f_{23}^2)$$

$\ldots \ldots \ldots \ldots$

Simple rescaling would not do! Use similar equations for duals:

$$x \hat{F}(m_{11} - \hat{f}_{11}) \equiv x \hat{F} f_{11} = y_1^2 (\hat{f}_{22}\hat{f}_{33} - \hat{f}_{23}^2)$$

and $$(\hat{f}_{22}\hat{f}_{33} - \hat{f}_{23}^2) = (m_{22} - f_{22})(m_{33} - f_{33}) - (m_{23} - f_{23})^2$$

Allows to express quadratic in $f$ terms through linear and $f$-indep. terms and $\hat{F}$
Resulting system:

\[ x (F + \hat{F}) f_{11} = y_1^2 \left[ (m_{22} m_{33} - m_{23}^2) - (m_{22} f_{33} + m_{33} f_{22} - 2m_{23} f_{23}) \right] + x F m_{11} \]

.......... 

Now one can rescale:

\[ f_{ij} = \lambda^{1/3} f'_{ij}, \quad m_{ij} = \lambda^{1/3} m'_{ij}, \quad y_{ij} = \lambda^{1/3} y'_{ij}, \]

Fix \( \lambda \) by requiring \( F' \equiv \det f' = 1 \)

Determinant condition: \( x^3 F \hat{F} = -y_1^2 y_2^2 y_3^2 \Rightarrow \)

\[ x^3 \hat{F}' = -(y_1' y_2' y_3')^2 \]

System of 6 linear eqs. for \( f'_{ij} \) – easily solved. Substitution into \( F' = 1 \)

\( (F = \lambda) \Rightarrow 8\text{th order characteristic eq. for } \lambda. \) Yields 4 pairs of dual solutions for the matrix \( f \).
Conclusions

- In LR - symmetric models (minimal LR, Pati-Salam, $SO(10)$, $E_6$, ...) one naturally has type I + II seesaw mechanism of neutrino mass generation.

- Discrete LR symmetry (parity) leads to a relation between type I and type II contributions to $m_\nu$, which results in a duality property of the seesaw formula: $f \Leftrightarrow m_\nu/v_L - f$.

- For given $y$, there are $2^n$ (8 for three lepton generations) matrices $f$ which result in exactly the same mass matrix of light neutrinos $m_\nu$.

- A simple analytic method developed for solving the seesaw nonlinear matrix equation. Allows bottom-up reconstruction of the Yukawa coupling matrix $f$ of heavy RH neutrinos.

- The results may be used for neutrino mass model building and studies of baryogenesis via leptogenesis. Allow to analytically explore the interplay of type I and II contributions to $m_\nu$. 