SECOND QUANTIZATION, ANOMALIES, AND GROUP EXTENSIONS

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CONTENTS:

1. BASIC FACTS ABOUT PSEUDODIFFERENTIAL OPERATORS

Let H be the Hilbert space of square-integrable functions $\psi : \mathbb{R}^n \to \mathbb{C}^N$. If $a(x)$ is a smooth bounded $N \times N$ matrix valued function on $\mathbb{R}^n$ we can define a linear operator $A : H \to H$ by $(A\psi)(x) = a(x)\psi(x)$. The same is true for Fourier

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transformed functions

$$\hat{\psi}(p) = \frac{1}{(2\pi)^{n/2}} \int e^{ip\cdot x}\psi(x)d^n x,$$

we can define $B : H \to H$ by

$$\widehat{(B\psi)}(p) = b(p)\hat{\psi}(p)$$

for a bounded smooth function $b(p)$ of the momentum p. Taking linear combinations of products of operators AB one is naturally lead to the following construction. Let $a(x,p)$ be a smooth function of coordinates x and momenta p. Then under suitable additional asymptotic conditions on the function $a(x,p)$ (to which we shall return below) the operator $A : H \to H$ is defined, at least in a dense subspace of H,

$$(A\psi)(x) = \frac{1}{(2\pi)^n} \int e^{-ix\cdot p} a(x,p) e^{iy\cdot p}\psi(y) d^n y\, d^n p. \tag{1.1}$$

This is a *pseudodifferential operator* with *a symbol* a. In the following we shall restrict the class of symbols to so-called classical symbols. Note that if

$$a(x,p) = \sum a_k(x) p_1^{k_1} \dots p_n^{k_n}$$

is a polynomial in the momenta then A is just a linear partial differential operator.

Definition 1.2. *Let Ω be an open subset of $\mathbb{R}^n$. The symbol class $S^m(\Omega)$, where m is an integer, consists of smooth (matrix valued) functions $a(x,p)$ on $\Omega \subset \mathbb{R}^n$ such that there exists a sequence $a_j(x,p)$ of smooth functions, $j = m, m-1, m-2, \dots,$ with the homogenuity properties*

$$a_j(x,\lambda p) = \lambda^j a(x,p) \text{ for } \lambda > 1 \text{ and } |p| > 1$$

and such that asymptotically

$$a(x,p) \sim \sum a_j(x,p).$$

The last condition means that

$$c(x,p) = a(x,p) - \sum_{i=0}^{K} a_{m-i}(x,p)$$

is in the class $S_1^{m-K-1}(\Omega)$ of functions c with the property

$$|D_x^\beta D_p^\alpha c(x,p)| \le C(W,\alpha,\beta)(1+|p|)^{m-K-1-|\alpha|}$$

in any compact subset $W \subset \Omega$. Here

$$D_p^\alpha = \frac{d^{\alpha_1}}{dp_1^{\alpha_1}} \cdots \frac{d^{\alpha_n}}{dp_n^{\alpha_n}} \text{ and } |\alpha| = \alpha_1 + \dots \alpha_n.$$

We shall study PSDO's defined by symbols in $S^m(\Omega)$. For example, all smooth symbols which are polynomials in the coordinates p_i are in $S^m(\Omega)$, where m is the degree of the polynomial. For this reason all partial differential operators which smooth coefficient functions are classical PSDO's. Any smooth rational function is in S^m for a suitable m. The function $\exp(-p^2)$ is in S^m for any m; we denote by $S^{-\infty}$ the intersection of all S^m's and we call operators defined by symbols in $S^{-\infty}$ *infinitely smoothing.* Note that if $a \in S^{-\infty}$ then we may choose all $a_j = 0$. Thus a the asymptotic expansion does not determine the symbol (and associated operator) uniquely. On the other hand:

Theorem 1.3. *Let $a_m, a_{m-1}, \dots$ be a sequence of symbols such that $a_j \in S^j(\Omega)$. Then there exists a symbol $a \in S^m(\Omega)$ such that*

$$a \sim \sum_{j \le m} a_j.$$

Proof. Choose compact sets $W_1 \subset W_2 \subset \cdots \subset \Omega$ such that $\Omega = \cup_j W_j$. Choose a smooth function $\phi : \mathbb{R}^n \to \mathbb{R}$ such that $\phi(p) = 0$ for $|p| \le 1/2$ and $\phi(p) = 1$ for $|p| \ge 1$. Define

$$a(x,p) = \sum_{j=0}^{\infty} \phi(\epsilon_j p) a_{m-j}(x,p),$$

where the ϵ_j's are small positive numbers such that

$$|D_x^\beta D_p^\alpha \phi(\epsilon_j p) a_{m-j}(x,p)| \le 2^{-j}(1+|p|)^{m-j+1-|\alpha|}$$

for $|\alpha| \le j - i$ and $x \in W_i$.

We state without a proof (for a proof see M. Taylor: *Pseudodifferential Operators,* Princeton University Press, 1981):

Theorem 1.4. *Let $a \in S^m$ and $b \in S^p$ and A, B the associated PSDO's. Then AB is represented by a symbol $c \in S^{m+p}$ with an asymptotic expansion*

$$c \sim a * b = \sum_{\alpha} \frac{(-i)^{|\alpha|}}{\alpha!} D_p^\alpha a(x,p) D_x^\alpha b(x,p),$$

where $\alpha! = \alpha_1! \dots \alpha_n!$ and the sum is over all sequencies of nonnegative integers.

The content of the theorem is easy to understand: Since p_j is the Fourier transform of the partial derivative operator $P_j = -i\frac{\partial}{\partial x_j}$ we have the Heisenberg commutation relations $[P_j, x_k] = -i\delta_{jk}$ and the product rule is just the rule for the rearrangement of the factors in the operator $a(x,P)b(x,P)$ as $c(x,P)$, coordinates on the left and momenta on the right.

We shall need some facts about traces of PSDO's. Let us assume for a moment that all functions have support in the box $0 \le x_i \le L$ in $\mathbb{R}^n$. We use the Fourier basis $\psi_k = \frac{1}{L^{n/2}} \exp(ik \cdot x)$ with $k = (k_1, \dots, k_n)$ and $Lk_i/2\pi \in \mathbb{Z}$. Then the diagonal matrix elements of a PSDO A, with a symbol a, are $< k|A|k >= \frac{1}{L^n} \int a(x,k) d^n x$. Thus the trace is given by

$$\mathrm{tr}\, A = \frac{1}{L^n} \sum_k \int_{0 \le x_i \le L} \mathrm{tr}\, a(x,k) d^n x,$$

where the trace under the integral is the $N \times N$ matrix trace of the symbol. In the limit $L \to \infty$ the discrete sum over k_i is replaced by an integral over continuous momenta p_i, with a measure $dp_i = 2\pi dk_i/L$, and the trace is

$$\mathrm{tr}\, A = \frac{1}{(2\pi)^n} \int \int \mathrm{tr}\, a(x,p) d^n x d^n p. \tag{1.5}$$

Assuming that the symbol, as a function of the x coordinate, is decreasing more rapidly than $1/|x|^n$ at the infinity, we conclude that the trace is absolutely converging if and only if $a \in S^m$ with $m < -n$. For example, in dimension $n = 1$ symbols of decree < -1 are trace-class whereas a symbol of decree -1 has typically a logarithmically diverging trace. Higher order symbols have polynomially diverging traces (as a function of a momentum space cut-off $|p| < \Lambda$.)

In quantum field theory a common PSDO is the Green's function of a partial differential operator. For example, the operator $-\Delta + k^2$ has a symbol $p^2 + k^2$ and its Green's function is $(p^2+k^2)^{-1}$. This is a symbol of degree -2. It has a finite trace (in a box of finite size) only in dimension $n=1$. When $n=2$ the trace is logarithmically diverging.

Any PSDO of degree zero which has bounded support as a function of x defines a bounded operator in the Hilbert space H. Denote

$$a_q(p) = (2\pi)^{-n/2}\int a(x,p)e^{ix\cdot q}\,d^n x.$$

Then

$$a(x,p) = (2\pi)^{-n/2}\int a_q(p)e^{-ix\cdot q}\,d^n q.$$

Because of

$$q^\alpha a_q(p) = (2\pi)^{-n/2}\int (iD_x)^\alpha a(x,p)e^{ix\cdot q}\,d^n x$$

we have

$$|a_q(p)| \leq C_j(1+|q|)^{-j}$$

for any positive integer j. It follows that $||a_q(D)u||_{L^2} \leq C_j(1+|q|)^{-j}||u||_{L^2}$. Because of $a(x,D) = (2\pi)^{-n/2}\int \exp(-ix\cdot q)\, a_q(D) d^n q$ and $|\exp(ix\cdot q)| = 1$ we get

$$||a(x,D)u||_{L^2} \leq C_j\int (1+|q|)^{-j} d^n q\, ||u||_{L^2} \leq C'||u||_{L^2}$$

for $j>n$.

In the following the sign operator ϵ of a Dirac operator in odd dimensions will play a central role. Recall that the Dirac operator (in an euclidean space $\mathbb{R}^n$) is

$$D = -i\sum_{k=1}^{n} \gamma^k\partial_k \tag{1.6}$$

where the γ_k's are complex $2^{(n-1)/2}\times 2^{(n-1)/2}$ matrices,

$$\gamma^k\gamma^j + \gamma^j\gamma^k = 2\delta_{jk}. \tag{1.7}$$

Thus the symbol of D is $\gamma^k p_k$ (Einstein summation convention). Because of (1.7) it has the property $(\gamma^k p_k)^2 = |p|^2$. The sign operator $\epsilon = D/|D|$ has symbol $\not p/|p|$; this has a singularity at $p=0$ and we interprete ϵ as the limit

$$\epsilon = \lim_{t\to 0+}\frac{\not p + it}{|p|+it}. \tag{1.8}$$

This means that the action of ϵ on zero modes of D is defined as a multiplication by i.

In the case $n=1$ the sign operator is simply $p/|p|$. Since the symbol is constant except at $p=0$ its derivative is zero for large $|p|$ and it follows that $[\epsilon, A]$ is an infinitely smoothing operator for any PSDO A. This is true in particular when A is a multiplication operator by a smooth function $a(x)$. Thus in the case of a finite box $[\epsilon, A]$ is a trace-class operator. This proves that the multiplication operators on a unit circle satisfy $[\epsilon, A] \in L_2$. (Functions on the unit circle are interpreted as functions in $[0, 2\pi] \subset \mathbb{R}$ with periodic boundary conditions.)

In higher dimensions the situation is different. The momentum space partial derivatives of the symbol ϵ are now symbols in S^{-1}. In n dimensions the only thing we can say about the commutator $[\epsilon, A]$, when A is a multiplication operator by matrix valued function $a(x)$ such that $[\gamma^k, a(x)] = 0$, is that it is an operator of degree -1 and therefore $|[\epsilon, A]|^j$ is trace-class for $j > n$. (We still assume that $a(x)$ has compact support.)

2. WODZICKI RESIDUE AND CENTRAL EXTENSIONS OF ALGEBRAS OF PSEUDODIFFERENTIAL OPERATORS

On the algebra of PSDO's there is (and essentially unique) trace functional. In dimension $d=1$ this is known as the Adler-Manin residue, [Ad,Ma], and is defined as follows. Consider a classical PSDO with a symbol

$$a(x,p) \sim a_n(x,p) + a_{n-1}(x,p) + \dots$$

with $a_k(x, \lambda p) = \lambda^k a_k(x,p)$ as usual, for $\lambda > 1$ and $|p| >> 0$. Note that in dimension one we have exactly two independent alternatives: either $a_k(x,p) = b(x)p^k$ or $a_k(x,p) = b(x)|p|^k$ for large momenta. The Adler-Manin residue was defined under the assumption that the first alternative is true and then

$$\mathrm{Res}(a) = \frac{1}{2\pi} \int \mathrm{tr}\, a_{-1}(x, p=1) dx.$$

Under this assumption we can write

$$a(x,p) \sim \sum_{k \leq n} b_k(x) p^k.$$

In general, the residue is defined either by

$$\text{Res}(a) = \frac{1}{2\pi} \int \sum_{p=\pm 1} \text{tr}\, a_{-1}(x,p) dx \tag{2.1}$$

or by

$$\text{Res}'(a) = \text{Res}(\epsilon\, a),$$

where $\epsilon = p/|p|$. This is specific to dimension $d = 1$. There are two linearly independent residues because the 'sphere' $|p| = 1$ in the momentum space has two disconnected components. In higher dimensions the sphere is connected and there is only one residue. By construction, the residue is a linear functional. It also satisfies the trace condition

$$\text{Res}(a * b - b * a) = 0. \tag{2.2}$$

This is seen by a simple integration by parts procedure.

We can define a central extension of the algebra of PSDO's on a circle by, [Ra, Kh-Kr],

$$c(a,b) = \frac{1}{2}\text{Res}[\ell, a] b, \tag{2.3}$$

where $\ell = \log(p)$ and, if not otherwise stated, the products of symbols are defined as star products. Note that ℓ is not a PSDO symbol but the commutator $[\ell, a]$ is a classical symbol for any PSDO a. The 2-cocycle property

$$c(a, [b,c]) + c(b, [c,a]) + c(c, [a,b]) = 0$$

follows from the fact that $\text{Res}[\ell, x] = 0$ for any PSDO x and from the trace property of Res. Actually, we have two different central extensions because of the two different residues; we have also

$$c'(a,b) = \frac{1}{2}\text{Res}'[\ell, a] b = \frac{1}{2}\text{Res}\, \epsilon[\ell, a] b. \tag{2.4}$$

If a, b are just functions of the coordinate x, then

$$c'(a,b) = \frac{-i}{2\pi} \int a'(x) b(x) \tag{2.5}$$

which we recognize as the central term of an affine Lie algebra.

In higher dimensions the definition of the residue is

$$\text{Res}(a) = \frac{1}{(2\pi)^d} \int dx \int_{|p|=1} \text{tr}\, a_{-d}(x,p) dp. \tag{2.6}$$

There is a more geometric definition: Let $\theta = p_i dx_i$ be the symplectic 1-form in the phase space so that $d\theta$ is the symplectic 2-form.

Exercise Show that the form $a_{-d}(x,p)\theta(d\theta)^{d-1}$ is closed outside the singularity at $p = 0$.

Let S be any surface with winding number equal to 1 around the point p in the momentum space. Then

$$\text{Res}(a) = \frac{1}{(2\pi)^d} \int dx \int_S \text{tr}\, a_{-d} \theta (d\theta)^{d-1}. \tag{2.7}$$

The value of the residue does not depend on the choice of S.

Remark The trace property $\text{Res}[a,b] = 0$ is in general guaranteed only on a compact manifold without boundary (even if the x-integration is assumed to converge). Of course, on a compact manifold one has to use local coordinates and patch together the residue formula from local pieces. On a noncompact manifold like $\mathbb{R}^d$ the residue still makes sense provided that we put enough strong vanishing condition on the symbols at infinity (in order to avoid boundary terms in integrations by parts); $|a(x,p)| \to 0$ at least like $|x|^{-d-\epsilon}$ is sufficient.

The ordinary trace of a PSDO defined by a symbol a is

$$\text{tr}(a) = \frac{1}{(2\pi)^d} \int \text{tr}\, a(x,p) dx\, dp \tag{2.8}$$

provided that the integral converges; this is the case if $deg(a) < -d$ and then the operator is said to be trace-class (the integral converges absolutely). If this is not the case we can define a cut-off trace

$$\text{tr}_\Lambda(a) = \frac{1}{(2\pi)^d} \int dx \int_{|p|<\Lambda} \text{tr}\, a(x,p) dp, \tag{2.9}$$

where $\Lambda > 0$ is a real parameter. If the degree n of a is an integer then asymptotically

$$\text{tr}_\Lambda(a) = \Lambda^{n+d} \alpha_{n+d} + \cdots + \Lambda \alpha_1 + \alpha_{log} \log(\Lambda) + \alpha_0(\Lambda), \tag{2.10}$$

where $\alpha_0(\Lambda)$ has a finite limit α_0 as $\Lambda \to \infty$. Note that the logarithmic piece is absent if the degree of a is not an integer; the logarithmic divergence is due to the term a_{-d} in the asymptotic expansion. Writing $a_{-d} = \frac{b(x,p)}{|p|^d}$, where $b(x,p)$ is scale invariant in momentum space, and performing first the radial integration and then the angular integration in momentum space, we observe that $\text{Res}(a)$ is equal to the coefficient α_{log} of the logarithmic divergence. This shows that the operator residue of an PSDO is equal to the *Dixmier trace.*

Note also that there is still another 'trace' we can define: $\text{TR}(a) = \alpha_0$. In general, $\text{TR}[a,b] \neq 0$, except when $deg(a+b) \neq -d$. In physical applications p has the dimension of momentum and therefore one has to replace $\log(\Lambda)$ by $\log(\Lambda/\lambda)$, where λ is some fixed scale parameter. The finite part α_0 will then depend on the choice of λ and one should write $\text{TR}_\lambda(a) = \lim_{\Lambda\to\infty} a_0(\lambda, \Lambda)$ as a reminder of the scale dependence of the generalized trace.

As already said, on the algebra of PSDO's in dimension $d > 1$ there is a unique residue and for this reason there is (up to a scale and coboundaries) only one central extension, defined by the 2-cocycle

$$c(a,b) = \frac{1}{2}\text{Res}[\ell, a]b,$$

where $\ell = \log(|p|)$. However, for certain subalgebras we have more choices. A cocycle which will be important in quantum field theory is defined as follows. First, let ϵ be a PSDO of degree zero such that $\epsilon^2 = 1$ and $[\ell, \epsilon] = 0$. We define $\mathcal{Z}_2$ as the algebra of bounded PSDO's a acting on sections of a fixed vector bundle over the physical space M of dimension d satisfying the condition $deg[\epsilon, a] < -d/2$. Note that on a compact manifold any PSDO of degree strictly less than $-d/2$ is Hilbert-Schmidt. The set $\mathcal{Z}_2$ is a subalgebra because the operators of degree less than n form an operator ideal in the space of bounded PSDO's. A PSDO is bounded iff its degree is nonpositive.

We define

$$c'(a,b) = \frac{1}{2}\text{Res}\,\epsilon[\ell, a]b. \tag{2.11}$$

Theorem 2.12. *The bilinear form c' on $\mathcal{Z}_2$ is a 2-cocycle.*

Proof. Denote
$$\mathrm{Res}'(a) = \mathrm{Res}(\epsilon\, a).$$

Since $\mathrm{Res}[a, b] = 0$ for any pair of PSDO's a, b we have

$$\begin{aligned}\mathrm{Res}\,\epsilon[\epsilon, a][\epsilon, b] &= \mathrm{Res}(a\epsilon b - \epsilon ab - ab\epsilon + \epsilon a\epsilon a\epsilon)\\ &= 2\mathrm{Res}(a\epsilon b - \epsilon ab) = -2\mathrm{Res}[\epsilon, a]b = -2\mathrm{Res}\epsilon[a, b].\end{aligned}$$

If now $deg[\epsilon, a] < -d/2$ and $deg[\epsilon, b] < -d/2$ then $deg\epsilon[\epsilon, a][\epsilon, b] < -d$ and it follows that the residue vanishes. Thus

$$\mathrm{Res}'[a, b] = 0 \tag{2.13}$$

for any $a,, b \in \mathcal{Z}_2$. Performing a partial integration in momentum space one gets $\mathrm{Res}[\ell, a] = 0$ for any PSDO a. Since ℓ commutes with ϵ, we have also

$$\mathrm{Res}'[\ell, a] = 0 \tag{2.14}$$

although ℓ is not a PSDO. Define

$$\omega(a, b, c) = c'(a, [b, c]) + c'(b, [c, a]) + c'(c, [a, b]).$$

Then

$$2\omega(a, b, c) = \mathrm{Res}'([\ell, a][b, c] + [\ell, b][c, a] + [\ell, c][a, b]). \tag{2.15}$$

Using (2.13) and (2.14) we get

$$\omega(a, b, c) = \mathrm{Res}'\left(a[[b, c], \ell] + a[[\ell, b], c] + a[[c, \ell], b]\right) = 0,$$

by Jacobi's identity, proving that c' is indeed a 2-cocycle.

3. CANONICAL QUANTIZATION IN A FERMIONIC FOCK SPACE. BOGOLIUBOV AUTOMORPHIMS AND CENTRAL EXTENSIONS

Let H be a complex Hilbert space. We consider an algebra with a unit 1, the algebra of *canonical anticommutation relations* (CAR), generated by elements $a(u), a^*(v)$ with $u, v \in H$, with the only relations

$$\begin{aligned} a^*(u)a(v) + a(v)a^*(u) &= (v,u) \\ a(u)a(v) + a(v)a(u) &= 0 \\ a^*(u)a^*(v) + a^*(v)a^*(u) &= 0 \end{aligned} \tag{3.1}$$

and the relations arising from the requirement that $u \mapsto a^*(u)$ is linear and $u \mapsto a(u)$ is antilinear. If H is finite-dimensional, $\dim H = n$, the CAR algebra has dimension 2^{2n}, otherwise it is infinite-dimensional.

Suppose $H = H_+ \oplus H_-$, where $H_\pm$ are closed subspaces. Let $\pi_\pm$ be the corresponding orthogonal projections. Assume that the elements of the CAR algebra are represented by linear operators in a Hilbert space $\mathcal{F}$ such that $a^*(u)$ is the adjoint of $a(u)$ for any $u \in H$. We say that this is a Fock representation with a Dirac vacuum $|0>$ if it is irreducible and there is a (normalized) vector $|0> \in \mathcal{F}$ such that

$$\begin{aligned} a(u)|0> &= 0 \text{ for all } u \in H_+ \\ a^*(v)|0> &= 0 \text{ for all } v \in H_-. \end{aligned} \tag{3.2}$$

The vector $|0>$ is sometimes called the 'Dirac sea'. One can prove that for each polarization $H = H_+ \oplus H_-$ there is a (up to equivalence) unique Fock representation. One has also the following theorem:

Theorem 3.3. *Two different polarizations $H = H_+ \oplus H_- = W_+ \oplus W_-$ define equivalent Fock representations of the CAR algebra if and only if the projections $W_+ \to H_-$ and $W_- \to H_+$ are Hilbert-Schmidt.*

We skip the proofs. Concerning the basic properties of representations of CAR algebra and references to original articles we refer to the review article [Ar]. Let

$g : H \to H$ be a unitary operator. A Bogoliubov automorphism of the CAR algebra is fixed by the conditions $a^*(v) \mapsto a^*(gv)$ and $a(v) \mapsto a(gv)$. A unitary map $T(g) : \mathcal{F} \to \mathcal{F}$ is a *implementor* of the Bogoliubov automorphism g if

$$T(g)a^*(v)T(g)^* = a^*(gv) \text{ and } T(g)a(v)T(g)^* = a(gv) \tag{3.4}$$

for all $v \in H$. Let us denote $\epsilon = \pi_+ - \pi_-$.

Theorem 3.5. *A Bogoliubov automorphism g is implementable if and only if $[\epsilon, g]$ is Hilbert-Schmidt.*

In a similar way, one defines Bogoliubov endomorphims and their implementors: now we are looking for antihermitean operators $dT(X) : \mathcal{F} \to \mathcal{F}$ such that

$$[dT(X), a^*(v)] = a^*(Xv) \text{ and } [dT(X), a(v)] = a(Xv) \tag{3.6}$$

where $X : H \to H$ is antihermitean. The condition for the existence of $dT(X)$ is that $[\epsilon, X]$ is Hilbert-Schmidt.

There is an explicite construction for operators $\hat{X} = dT(X)$ satisfying (3.6). Let $\{e_n\}$ be a basis of H such that $e_n \in H_+$ for $n \geq 0$ and $e_n \in H_-$ for $n < 0$. Denote $a_n = a(e_n)$ and $a_n^* = a^*(e_n)$. Define a normal ordering by

$$: a_n^* a_m := \begin{cases} -a_m a_n^* \text{ for } n = m < 0 \\ a_n^* a_m \text{ otherwise} \end{cases}$$

and set

$$\hat{X} = \sum X_{nm} : a_n^* a_m : \tag{3.7}$$

where $X_{nm} = (e_n, X e_m)$. The condition (3.6) is checked by a direct computation from the defining relations (3.1). Note that for finite matrices X

$$\hat{X} = \sum X_{nm} a_n^* a_m - \sum_{k<0} X_{kk}. \tag{3.8}$$

From this we get, for finite matrices,

$$[\hat{X}, \hat{Y}] = \sum X_{nm} Y_{kl} [a_n^* a_m, a_k^* a_l] = \sum [X, Y]_{ij} a_i^* a_j = \widehat{[X, Y]} + \sum_{k<0} [X, Y]_{kk}.$$

The constant term can be written as

$$c(X,Y) = \mathrm{tr}\, \pi_+[X,Y] = \frac{1}{4}\mathrm{tr}\, \epsilon[\epsilon, X][\epsilon, Y]. \tag{3.9}$$

The last form is continuous in the space of matrices $X, Y \in \mathbf{g}l_{res}$. Here the restricted linear Lie algebra $\mathbf{g}l_{res}$ consists of bounded operators X such that $[\epsilon, X]$ is Hilbert-Schmidt. The topology is defined by the HS norm on the off-diagonal blocks and by the operator norm on the diagonal blocks. Using the continuity of c one can show that

$$[\hat{X}, \hat{Y}] = \widehat{[X,Y]} + c(X,Y) \tag{3.10}$$

for all $X, Y \in \mathbf{g}l_{res}$, [Lu].

Unfortunately, the HS condition is seldom satisfied for operators of physical interest. The important cases where the condition is satisfied are local currents in $1+1$ -dimensional field theory models and external field scattering operators in all dimensions.

Example Let X be a multiplication operator $\psi \mapsto X(x)\psi(x)$ acting on vector valued functions ψ on the circle and let H be the Hilbert space of square-integrable functions ψ. Let $\epsilon = p/|p|$, the sign of the momentum operator on the circle. If X is smooth the symbol of the commutator $[\epsilon, X]$ is localized at $p = 0$ in the momentum space and therefore defines a HS operator. In this case $c(X,Y)$ can be explicitly evaluated (Exercise!) and the result is

$$c(X,Y) = \frac{1}{2\pi}\int \mathrm{tr}\, X'(x)Y(x)dx.$$

Note that this is, modulo the sign, the same as the cocycle c' in Section 2. There is a direct way to see that these should be the same. Using $\epsilon^2 = 1$ and the conditional trace $\mathrm{tr}_C(X) = \frac{1}{2}\mathrm{tr}(X + \epsilon X \epsilon)$ we first get

$$\frac{1}{4}\mathrm{tr}\, \epsilon[\epsilon, X][\epsilon, Y] = \frac{1}{2}\mathrm{tr}_C X[\epsilon, Y] = \frac{1}{2}\mathrm{tr}_C([X\epsilon, Y] - \epsilon[X,Y]).$$

The two terms on the right are not (even conditionally) trace-class but the divergences cancel and therefore we may write

$$c(X,Y) = \frac{1}{2}\mathrm{TR}\,[X\epsilon, Y] - \frac{1}{2}\mathrm{TR}\,\epsilon[X,Y].$$

The second term is a coboundary in the Hochschild cohomology and thus

$$c(X,Y) \sim \frac{1}{2}\mathrm{TR}\,[X\epsilon, Y].$$

For a trace of a commutator of PSDO's there are local explicit expressions, [Wo, CFNW]. One can show that

$$\mathrm{TR}[a,b] = -\frac{1}{2}\mathrm{Res}[\ell, a]b \tag{3.11}$$

which gives immediately the result we want.

In higher dimensions local multiplication operators satisfy a weaker condition. Let $\epsilon = \not{p}/|p|$ be the sign of a 'free' Dirac operator on a d-dimensional compact manifold. Assuming that X commutes with the Dirac gamma matrices, the commutator $[\epsilon, X]$ has then the principal symbol

$$[\epsilon, X] = -i(\frac{\gamma_j}{|p|} - \frac{\not{p}p_j}{|p|^3})\partial_j X + ...$$

which is a PSDO of order -1. This raised to power $d+\delta$ for any positive δ is trace-class and therefore $[\epsilon, X] \in L_{d+}$, where L_p is the Schatten ideal of operators X such that $|X|^p$ is trace-class. We define the Lie algebras $\mathbf{g}l_p$ as the Lie algebra of bounded operators for which $[\epsilon, X] \in L_{2p}$. In particular, $\mathbf{g}l_{res} = \mathbf{g}l_1$. We have an infinite chain of Lie algebras $\mathbf{g}l_1 \subset \mathbf{g}l_2 \subset \mathbf{g}l_3 \subset \dots \mathbf{g}l_\infty$ with $\mathbf{g}l_\infty$ the Lie algebra of operators with $[\epsilon, X]$ compact.

For example, in dimension $d=3$, $X \in \mathbf{g}l_{res}$ only when X is a constant. However, after subtracting a tail which behaves like $1/|p|$ for large momenta, a local multiplication operator X becomes a nonlocal operator which belongs to $\mathbf{g}l_{res}$. For example, when $d=3$ an a Riemannian manifold we can set

$$\tilde{X} = X + \frac{i}{4|p|^2}[\gamma^k, \gamma^l]p_k\partial_l X \tag{3.12}$$

and by a direct computation, using the defining property of Dirac matrices

$$\gamma_k\gamma_l + \gamma_l\gamma_k = 2\delta_{kl},$$

one sees that in the commutator $[\epsilon, \tilde{X}]$ all the terms which are of order $1/|p|$ cancel and the commutator is an operator of order -2. But in three dimensions such an operator is Hilbert-Schmidt and thus $\tilde{X} \in \mathbf{g}l_{res}$.

Next we shall explain a systematic way how to produce operators like $\tilde{X}$ satisfying the HS condition and the current algebra anomalies to which this kind of renormalization leads.

4. RENORMALIZATION IN EXTERNAL FIELD PROBLEMS AND CURRENT ALGEBRA

We shall study massless Dirac fermions coupled to a gauge potential A in Minkowski space. The potential is a smooth 1-form $A_\mu(x)\,dx^\mu$ in space-time with values in the Lie algebra $\mathbf{g}$ of a compact gauge group G. The elements of $\mathbf{g}$ are represented by hermitean (according to physics literature convention) matrices in the complex vector space $\mathbb{C}^N$. The free Dirac operator is then $i\gamma^\mu\partial_\mu$. The metric is $x^\mu x_\mu = x_0^2 - x_1^2 - \cdots - x_d^2$. The Dirac gamma matrices satisfy $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}$, γ_0 is hermitean and γ_k is antihermitean for $k \neq 0$. We fix a hermitean matrix Γ such that $\Gamma^2 = 1$ and $\Gamma\gamma_\mu = -\gamma_\mu\Gamma$. The chiral projectors are $P_\pm = \frac{1}{2}(\Gamma \pm 1)$.

The Dirac hamiltonian in background gauge field A is

$$D_A = -\gamma^0\gamma^k(i\partial_k + A_k) - A_0.$$

We shall assume that $A(x)$ and its derivatives vanish faster than $|x|^{-d/2}$ when $|x| \to \infty$.

The one-particle scattering operator S is defined as the limit of the time evolution operator in the interaction picture, $U_I(t,-t) \to S$ as $t \to \infty$. The time evolution in the Schrödinger picture is defined by

$$i\partial_t U(t,t_0) = h(t)U(t,t_0) \text{ with } U(t_0,t_0) = 1, \tag{4.1}$$

and in the interaction picture by

$$i\partial_t U_I(t,t_0) = V_I(t)U_I(t,t_0), \text{ with } U_I(t_0,t_0) = 1. \tag{4.2}$$

The interaction is $V_I(t) = e^{ith_0}V(t)e^{-ith_0}$, the total hamiltonian being $h(t) = D_A = h_0 + V(t)$ with $h_0 = D_0 = -i\gamma_0\gamma^k\partial_k$. The quantum divergences are related to the fact that when $V = -\gamma^0\gamma^k A_k(\mathbf{x},t) - A_0(\mathbf{x},t)$ is the interaction with a Yang-Mills potential then the quantization of $\hat{U}_I(t,-\infty)$ for intermediate times $t<\infty$ is not well-defined. Let $\epsilon = h_0/|h_0|$ be the sign of the free hamiltonian. Generically, the Hilbert-Schmidt property for $[\epsilon, V(t)]$; this makes the quantization of the time evolution problematic.

The following renormalization makes U_I quantizable, [M1, M2, LM]. For each (smooth) potential $A = A_0 dt + A_k dx^k$ one defines a unitary operator T_A in the 1-particle space with the following property. Define $U_{ren}(t,t_0) = T_A(t) U_I(t,t_0) T_A(t_0)^{-1}$. Then 1) $[\epsilon, U_{ren}(t,t_0)]$ is Hilbert-Schmidt, 2) $T_A(t)\to 1$ as $t\to\pm\infty$. The last property quarantees that the renormalization does not affect the scattering matrix S whereas the first condition guarantees that the renormalized time evolution is quantizable in the free Fock space.

The Hilbert-Schmidt condition on operators $[\epsilon, g]$ defines the restricted unitary group $\mathcal{U}_1 = \mathcal{U}_{res}$ of unitary operators g, [PrSe]. The second quantization of elements in $\mathcal{U}_1$ defines a central extension $\hat{\mathcal{U}}_1$ as discussed in detail in [PrSe]. Now we have a smooth path of operators $g(t) = U_{ren}(t) = U_{ren}(t,-\infty)$ in $\mathcal{U}_1$ with the initial condition $g(-\infty)=1$ and $g(+\infty)=S$.

The operator T_A is not uniquely defined. It is more convenient to define the transformation first in the Schrödinger picture (4.1). A simple formula which works is (here $\not{A} = \gamma^0\gamma^k A_k$ and $\not{E} = \partial_t \not{A} - \gamma_0\gamma_k\partial_k A_0 + i[A_0, \not{A}]$)

$$T_A = \exp\left(\frac14 [\frac{1}{D_0}, \not{A}] - \frac18 [\frac{1}{D_0}\not{A}\frac{1}{D_0}, \not{A}] - \frac{i}{4}\frac{1}{D_0}\not{E}\frac{1}{D_0}\right), \tag{4.3}$$

where it is understood that the singularity at the zero modes of D_0 is taken care of by an infrared regularization, for example $\frac{1}{D_0}\to\frac{D_0}{D_0^2+\alpha^2}$ for some nonzero real number α. In the interaction picture one uses the operator $\exp(ith_0)T_A\exp(-ith_0)$. Note that this choice commutes with the chiral projection operators $P_\pm$.

For the proof of validity of the choice (4.3) it is convenient to use the symbol calculus for pseudodifferential operators.

The time evolution equation for $U_{ren}(t,t_0) = T_A(t)U(t,t_0)T_A(t_0)^{-1}$ in the Schrödinger picture is

$$i\partial_t U_{ren}(t,t_0) = (h_0 + W(t))U_{ren}(t,t_0)$$

with

$$W(t) = (i\partial_t T_A)T_A^{-1} + T_A(h_0+V(t))T_A^{-1} - h_0. \tag{4.4}$$

Expanding the exponential and arranging terms according to powers of the inverse of momentum (i.e., of D_0,) one gets $[\epsilon, W] = R_1 + R_2 + \dots$, where the dots denote terms which behave explicitly as $|D_0|^k$ with $k \leq -2$ for high momenta, and

$$R_1 = \frac{1}{2}\frac{D_0}{|D_0|}\not{A} - \frac{1}{2}\not{A}\frac{D_0}{|D_0|} + \frac{1}{4}|D_0|\not{A}\frac{1}{D_0} - \frac{1}{4}D_0\not{A}\frac{1}{|D_0|} + \frac{1}{4}\frac{1}{|D_0|}\not{A}D_0 - \frac{1}{4}\frac{1}{D_0}\not{A}|D_0|$$

and the second term, which is quadratic in $\not{A}$, is

$$\begin{aligned} R_2 =& \frac{1}{4}\frac{1}{D_0}[|D_0|, \not{A}^2]\frac{1}{D_0} + \frac{1}{4}\left[\frac{1}{|D_0|}, \not{A}^2\right] + \frac{1}{8}D_0\left[\not{A}\frac{1}{D_0}\not{A}, \frac{1}{|D_0|}\right] + \\ &+ \frac{1}{8}\left[\not{A}\frac{1}{D_0}\not{A}, \frac{1}{|D_0|}\right]D_0 + \frac{1}{8}\frac{1}{D_0}\left[\not{A}\frac{1}{D_0}\not{A}, |D_0|\right] + \frac{1}{8}\left[\frac{1}{D_0}\not{A}\frac{1}{D_0}, |D_0|\right]\frac{1}{D_0}. \end{aligned}$$

Since the commutator $[|D_0|^k, \not{A}]$ is of order $|D_0|^{k-1}$ in momenta, all terms in R_2 are actually of order -2 or less. In order to estimate R_1 we write it in equivalent form

$$R_1 = \frac{1}{4}\frac{D_0}{|D_0|}\left[[\not{A}, |D_0|], \frac{1}{|D_0|}\right] - \frac{1}{4}\left[[\not{A}, |D_0|], \frac{1}{|D_0|}\right]\frac{D_0}{|D_0|}$$

and observe both terms are of order -2 for the same reason as in the case of R_2. We have disregarded all the low order terms because in three space dimensions the condition that a PSDO is Hilbert-Schmidt (which was required for canonical quantization) is precisely the requirement that the operator vanishes for high momenta faster than $|p| = |D_0|$ raised to power $-3/2$. Thus after our renormalization (= conjugation by the time-dependent unitary operator T_A) the gauge interaction can be lifted to a finite operator in the fermionic Fock space.

The total renormalized hamiltonian is now a sum of the free (unbounded) self-adjoint hamiltonian and the bounded self-adjoint interaction. By the Kato-Rellich theorem the total hamiltonian is self-adjoint and according to Stone's theorem it defines the time evolution as a strongly continuous unitary one-parameter group in the Fock space.

Current algebra

Let us next consider the effect of gauge transformations on states in the Fock space. In $1+1$ dimensions, and for chiral fermions, we already know that the quantization introduces a central extension (affine Kac-Moody algebra) in the Lie algebra of the group of gauge transformations LG. In higher dimensions there is also an extension of the group $Map(M,G)$ but which is more complicated, in particular, the fiber of the extension is an infinite-dimensional abelian group. With $'Map'$ we shall always mean smooth functions and we either assume that M is compact or $M=\mathbb{R}^d$ and the fields and their derivatives decrease at least like $1/|x|^{d+\delta}$ at infinity.

We shall work in the hamiltonian formulation: all fields are defined at a fixed time $t=0$. Let us denote by $\mathcal{A}$ the space of smooth vector potentials in the physical space (we shall consider in detail the example $\dim M=3$ below).

In the 'one-particle representation' the Schrödinger wave functions $\phi(A)$ are functions of $A\in\mathcal{A}$ taking values in the fermionic Hilbert space H. A gauge transformation $g\in Map(M,G)$ is acting through $\psi'(A)=g\cdot\psi(g^{-1}\cdot A)$ on a wave function; here $g\cdot A=gAg^{-1}+gd(g^{-1})$. Infinitesimally, we have the so-called Gauss law generators

$$(G_X\phi)(A)=X\phi(A)+\mathcal{L}_X\phi(A),$$

where the Lie derivative is defined as $\mathcal{L}_X\phi=\frac{d}{dt}\phi(e^{-tX}\cdot A)|_{t=0}$ with $X\in Map(M,\mathbf{g})$. The Gauss law generators satisfy the algebra $[G_X,G_Y]=G_{[X,Y]}$.

The renormalization by the A-dependent unitary operator T_A conjugates the Gauss law generators to

$$\tilde{G}_X=T_A^{-1}G_XT_A=T_A^{-1}XT_A+T_A^{-1}\mathcal{L}_XT_A+\mathcal{L}_X=\theta(X;A)+\mathcal{L}_X. \tag{4.5}$$

Note that

$$\begin{aligned}[]
&[\theta(X;A)+\mathcal{L}_X,\theta(Y;A)+\mathcal{L}_Y]\\
&\qquad=[\theta(X;A),\theta(Y;A)]+\mathcal{L}_X\theta(Y;A)-\mathcal{L}_Y\theta(X;A)+[\mathcal{L}_X,\mathcal{L}_Y]\\
&\qquad=\theta([X,Y];A)+\mathcal{L}_{[X,Y]}
\end{aligned} \tag{4.6}$$

That is, the functions $\theta(X;\cdot)$ form a 1-cocycle for the gauge action of $Map(M,\mathbf{g})$.

The quantum wave function is a function $\psi(A)$ taking values in the Fock space $\mathcal{F}$.

Proposisition. *Let T_A be defined by (4.3). Then $\theta(X;A)$ is in the Lie algebra of $\mathcal{U}_{res}$ for any $X\in Map(M,G)$.*

Proof: Since $D'_A = T_A^{-1}(h_0+V)T_A = h_0 + W_A$ and $[\epsilon, W_A]$ is HS, we observe that the sign $\epsilon'(A)$ of the Hamiltonian D'_A differs from ϵ by a HS operator. Let g be a finite gauge transformation.

$$(g\cdot\phi)(A) = (T_{g\cdot A}^{-1} g T_A)\phi(g^{-1}\cdot A) \equiv \omega(g;A)\phi(g^{-1}\cdot A).$$

Thus

$$\begin{aligned}[\epsilon, \omega(g;A)] &= \epsilon T_{g\cdot A}^{-1} g T_A - T_{g\cdot A}^{-1} g T_A \epsilon \\ &= (T_{g\cdot A}^{-1}\epsilon T_{g\cdot A}) g T_A - T_{g\cdot A}^{-1} g (T_A \epsilon T_A^{-1}) T_A \\ &= T_{g\cdot A}^{-1}\left((T_{g\cdot A}\epsilon T_{g\cdot A}^{-1})g - g(T_A\epsilon T_A^{-1})\right)T_A \\ &\equiv T_{g\cdot A}^{-1}\left(\epsilon(g\cdot A)g - g\epsilon(A)\right)T_A\end{aligned}$$

where we have used the equivariantness of the family of Dirac operators,

$$gD_A g^{-1} = D_{g\cdot A}.$$

Thus finite gauge transformations satisfy the HS condition; considering one- parameter subgroups one proves the HS condition for the generators.

The asymptotic expansion for θ is

$$\theta(X;A) = X + \frac{i}{4}\frac{[\not{p}, \not{\partial}X]}{|p|^2} + \theta_{-2} + O(-3) \tag{4.7}$$

with

$$\begin{aligned}\theta_{-2} = &-\frac{1}{4}\frac{[\gamma_k, \not{A}]}{|p|^2}\partial_k X + \frac{1}{2}\frac{[\not{p}, \not{A}]}{|p|^4} p_k \partial_k X \\ &+ \frac{1}{16}\frac{[\not{p}, \not{A}]}{|p|^4}[\not{p}, \not{\partial}X].\end{aligned}$$

In the abobe formula the commutators are only finite-dimensional matrix (not star product) commutators. If X is any bounded bilinear quantity in the fermion creation and annihilation operators such that its off-diagonal blocks in the one-particle representation with respect to the energy polarization are HS, then the second quantized operator $\hat{X}$ is well-defined (after normal ordering) and, repeating (3.10),

$$[\hat{X}, \hat{Y}] = \widehat{[X,Y]} + c(X,Y), \tag{4.8}$$

where c is a Schwinger term, [Lu],

$$c(Y,Y) = \frac{1}{4}\mathrm{tr}\, \epsilon[\epsilon, X][\epsilon, Y]. \tag{4.9}$$

In the $3+1$ dimensional case one just inserts from (4.5) to (4.9):

$$c(X,Y;A) = \frac{1}{4}\mathrm{tr}\, \epsilon[\epsilon, \theta(X;A)][\epsilon, \theta(Y;A)]. \tag{4.10}$$

Using the equivalence of the cocycle (4.10) with the local form, written in terms of the residue, one gets

$$c(X,Y;A) \sim c_{loc} = \frac{1}{2}\mathrm{Res}\, \epsilon[\ell, \theta(X;A)]\theta(Y;A).$$

This form of the cocycle can be computed more explicitly since the residue (in three dimensions) depends only on the degree = -3 part of the operator and each momentum space differentiation decreases the degree by one unit, thus the residue is an integral of a local differential polynomial. In the case of chiral (2-component) fermions one gets, after a straightforward computation,

$$c_{loc}(X,Y;A) = \frac{i}{24\pi^2}\int_x \mathrm{tr}\, A[dX, dY], \tag{4.11}$$

which is the cocycle derived in [M3, F-Sh] in a different context.

5. THE PHASE OF THE QUANTUM SCATTERING MATRIX. INFINITE-DIMENSIONAL GROUPS AND THE LAMB SHIFT

Let $\hat G$ be a central extension of a Lie group G by $\mathbb{C}^\times$. The Lie algebra $\hat{\mathbf{g}}$ of $\hat G$ is a vector space direct sum $\mathbf{g} \oplus \mathbb{C}$. Let π be the projection on the second summand and let $\theta = g^{-1}dg$ be the left Maurer-Cartan one-form. We can then define a complex valued one-form ϕ on $\hat G$ by $\phi = \pi(\theta)$. This is a connection form in the principal $\mathbb{C}^\times$ bundle $\hat G \to G$. Its curvature is a left invariant two-form on G given by $\omega(X,Y) = c(X,Y)$, where left invariant vector fields X, Y on G are identified as elements of the Lie algebra and c is the 2-cocycle on $\mathbf{g}$ defining the central extension,

$$[(X,\lambda),(Y,\mu)] = ([X,Y], c(X,Y)). \tag{5.1}$$

Denote by GL_1 the group of invertible linear transformations $g : H \to H$ such that $[\epsilon, g]$ is Hilbert-Schmidt so that $\mathcal{U}_1$ is its unitary subgroup. Let us apply the above remarks to $G = GL_1$, and to the Lie algebra cocycle c arising when promoting the one-particle operators to operators in the fermionic Fock space, as discussed in the previous sections.

The central extension $\widehat{GL_1}$ is a nontrivial $\mathbb{C}^\times$ bundle over the base GL_1, [PrSe]. The elements of the group $\widehat{GL_1}$ (containing the unitary subgroup $\hat{\mathcal{U}}_1$) can be thought of equivalence classes of pairs (g, q), where $g \in GL_1$ and $q : H_+ \to H_+$ is an invertible operator such that $a - q$ is a trace-class operator,

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{5.2}$$

We have assumed that $\operatorname{ind} a = 0$. If this is not the case, the subspace H_+ must be either enlarged or made smaller by a suitable finite-dimensional subspace in order to achieve $\operatorname{ind} a = 0$. The equivalence relation is determined by $(g, q) \sim (g', q')$ if $g = g'$ and $\det(q' q^{-1}) = 1$. Thus the fiber of the extension is $\mathbb{C}^\times$ and it is parameterized by (the nonexisting) determinant of q.

The product is defined simply $(g, q)(g', q') = (gg', qq')$. Near the unit element in GL_1 we can define a local section $g \mapsto (g, a)$, [PrSe]. Denoting

$$g^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

we can write the connection form as

$$\phi_g = \operatorname{tr}\left[(g^{-1}dg)_a - q^{-1}dq\right] = \operatorname{tr}[\alpha da + \beta dc - q^{-1}dq]. \tag{5.3}$$

The curvature of this connection at $g = 1$ is

$$\omega = -\operatorname{tr}(db dc). \tag{5.4}$$

Interpreting the tangent vectors at $g = 1$ as elements in the Lie algebra we obtain

$$\omega(X, Y) = -\operatorname{tr}(b(X)c(Y) - b(Y)c(X)) = \frac{1}{4}\operatorname{tr}\, \epsilon[\epsilon, X][\epsilon, Y].$$

Thus the curvature of the connection is directly given through the Lie algebra central extension as promised.

We compute the parallel transport determined by the connection in the range of the local section. Let $g(t)$ be a path in GL_1, $-T \leq t \leq T$, with $g(-T) = 1$. The lift $(g(t), q(t))$ is parallel if

$$0 = \phi_{g(t),q(t)}(dg, dq) = \mathrm{tr}[\alpha(t)a'(t) + \beta(t)c'(t) - q(t)^{-1}q'(t)].$$

Thus the parallel transport, relative to the trivialization $g \mapsto (g, a)$, along the path $g(t)$ in the base is accompanied with the multiplication by the complex number

$$\exp\{-\int_{-T}^{T} \mathrm{tr}[(\alpha(t) - a(t)^{-1})a'(t) + \beta(t)c'(t)]dt\} \tag{5.5}$$

in the fiber $\mathbb{C}$.

Formally,

$$\mathrm{tr}\, q^{-1}q' = \mathrm{tr}[\alpha a' + \beta c']$$

and so

$$\det q(T) = \exp \int_{-T}^{T} \mathrm{tr}[\alpha(t)a'(t) + \beta(t)c'(t)]dt$$

and also

$$\det a(T) = \exp \int_{-T}^{T} \mathrm{tr}\, a(t)^{-1}a'(t)dt.$$

Individually, the traces in these two expressions do not converge, but putted together the trace converges and gives

$$\det(a(T)q(T)^{-1}) = \exp\{\int_{-T}^{T} \mathrm{tr}[(\alpha - a^{-1})a' + \beta c']dt\}. \tag{5.6}$$

Note that the exponent diverges outside of the domain of the local section, reflecting the fact that $\det a(T) = 0$ outside of the domain.

We can now apply the above results to the 'renormalized' one-particle time evolution operators $g(t) = U_{ren}(t)$ in the interaction picture. Let us, for the sake of simplicity, assume that the interaction is switched off outside of a finite interval $[-T, T]$ in time. Thus the 1-particle scattering operator is $S_A = U_{ren}(T) = U(T)$. For all times t, $g(t) \in \mathcal{U}_1$. On the other hand, in the Fock representation of $\widehat{GL_1}$ these correspond to elements $\hat{g}(t)$ in the central extension $\hat{\mathcal{U}}_1$. The phase of the quantum time evolution operator is then uniquely given by the parallel transport described above.

The Minkowskian effective action $Z(A)$ is by definition the vacuum expectation value of the quantum scattering operator $\hat{S}_A$. The vacuum is invariant under the free time evolution $\exp(itD_0)$ and taking into account the assumption that the interaction has essentially compact support in time, we can write

$$Z(A) =< 0|(g(T), q(T))|0 > . \tag{5.7}$$

The vacuum expectation value is given by a simple formula, [PrSe, M4],

$$< 0|(g, q)|0 >= \det(aq^{-1}) \tag{5.8}$$

and therefore the parallel transport (with respect to the given local trivialization) is equal the effective action $Z(A)$.

Let us compute $\log(Z)$ to the lowest order in the interaction A in the case of QED, i.e., massive fermion coupled to a Maxwell potential. Because of the unitarity relation $a^*a+c^*c = 1$ the inverse $a^{-1} \equiv a^*$ modulo terms of order A^2 since the the off diagonal blocks of the time evolution $g(t)$ must contain the potential at least to order one. For this reason the term $(\alpha - a^{-1})a'$ in the phase does not give contributions to the order A^2. On the other hand, because of unitarity, $\beta c' = c^*c' \equiv -bc'$ modulo terms of order higher than two.

Next we use the Dyson expansion

$$g(t) = 1 - i \int_{-\infty}^{t} V_I(s)ds + (-i)^2 \int_{t>s_1>s_2} V_I(s_1)V_I(s_2)ds_1 ds_2 + \dots \tag{5.9}$$

for the time evolution operator in the interaction picture. This gives the lowest (A^2 term) for $\log(Z)$,

$$\log(Z) = \int_{s>t} \mathrm{tr}\ \pi_+ V_I(s) \pi_- V_I(t) \pi_+ dt\, ds.$$

We shall use the integral representation

$$\begin{aligned} &\frac{1}{2\pi i} \int \mathrm{tr} \frac{\not{p} - m}{p^2 - m^2 + i\epsilon} e^{ip_0 T} dp_0 \\ &= \gamma_0 [\theta(T)\theta(h_0(\mathbf{p}))e^{-iT\omega_p} - \theta(-T)\theta(-h_0(\mathbf{p}))e^{iT\omega_p}] \end{aligned} \tag{5.10}$$

where $\omega_p = \sqrt{\mathbf{p}^2 + m^2}$, $\theta(x) = 1$ for $x \geq 0$ and $\theta(x) = 0$ for $x < 0$ and $h_0(\mathbf{p}) = \gamma^0\gamma^k p_k + \gamma^0 m$ is the momentum representation for the free Dirac hamiltonian. In the usual QED perturbation theory the second order effect is given by the (diverging) Feynman integral

$$\frac{1}{4\pi}\int \mathrm{tr}\frac{\not p - m}{p^2-m^2+i\epsilon}\hat{\not A}(p-q)\frac{\not q - m}{q^2-m^2+i\epsilon}\hat{\not A}(q-p)d^4p\,d^4q. \tag{5.11}$$

By a Fourier transform of the potential in the time variable this integral can be written as

$$\frac{1}{8\pi^2}\int \mathrm{tr}\frac{\not p - m}{p^2-m^2+i\epsilon}e^{is(p_0-q_0)}\not A(s,\mathbf{p}-\mathbf{q})\frac{\not q - m}{q^2-m^2+i\epsilon}e^{it(q_0-p_0)}\not A(t,\mathbf{q}-\mathbf{p})ds,dt\,d^4p\,d^4q.$$

Using the trick (5.10) we can write the integral as

$$\begin{aligned}-\frac{1}{2}\int &\mathrm{tr}[\theta(s-t)\theta(h_0(\mathbf{p}))e^{-i(s-t)h_0(\mathbf{p})}-\theta(t-s)\theta(-h_0(\mathbf{p}))e^{-ih_0(\mathbf{p})(s-t)}]\\ &\times \not A(s,\mathbf{p}-\mathbf{q})[\theta(t-s)\theta(h_0(\mathbf{q}))e^{-i(t-s)h_0(\mathbf{q})}-\theta(s-t)\theta(-h_0(\mathbf{q}))e^{-i(t-s)h_0(\mathbf{q})}]\\ &\times \not A(t,\mathbf{q}-\mathbf{p})e^{-ih_0(\mathbf{q})(t-s)}ds\,dt\,d^3\mathbf{p}\,d^3\mathbf{q}.\end{aligned}$$

Since $\theta(T)\theta(-T)=0$ and $\theta(T)^2=\theta(T)$ we get

$$\int \mathrm{tr}\,\theta(s-t)\pi_+V_I(t)\pi_-V_I(s)\pi_+ds\,dt$$

which is exactly the second order term in our geometric definition of $\log(Z)$. Note that this discussion is formal in the sense that diverging Feynman integrals are involved. However, we may apply some renormalization method (for example, the family of T_A operators described earlier, to bring the time evolution to the group $\mathcal{U}_1$) in order to make sense of these integrals.

More on infinite-dimensional groups and QED, see [GB-V]; on the connection of CAR representations and determinant bundles, [PrSe, M4, SW].

6. THE BUNDLE GERBE APPROACH TO HAMILTONIAN ANOMALIES. THE DIXMIER-DOUADY CLASS

In this section we want to describe a more geometric approach to the problem of construction of the family of Fock spaces parametrized by external vector potentials and the action of the gauge group. Actually, the method here is very general and applies as well to the case of an external metric field or any other interactions for that matter. In order to keep the discussion as simple as possible we shall restrict to the case of vector potentials.

The fermionic Fock spaces parametrized by Yang-Mills potentials form a vector bundle $\mathcal{F}$ over the space $\mathcal{A}$. In the case of chiral massless fermions there are subtleties in defining this bundle. The difficulty is related to the fact that the splitting of the one particle fermionic Hilbert space H to positive and negative energies is not a continuous function of the external field. One can easily construct paths in the space of external fields such that at some point on the path a positive energy state dives into the negative energy space (or vice versa). These points are obviously discontinuities in the definition of the space of negative energy states and therefore the fermionic vacua do not form of smooth vector bundle over the space of external fields. This problem does not arise if we have massive fermions in the temporal gauge $A_0 = 0$. In that case there is a mass gap $[-m, m]$ in the spectrum of the Dirac hamiltonians and the polarization to positive and negative energy subspaces is indeed continuous.

If λ is a real number not in the spectrum of the hamiltonian then one can define a bundle of fermionic Fock spaces $\mathcal{F}'_{A,\lambda}$ over the set U_λ of external fields A, $\lambda \notin Spec(D_A)$. The vacuum in $\mathcal{F}'_{A,\lambda}$ is defined by the polarization of the one-particle space to positive and negative spectrum of the operator $D_A - \lambda$. It turns out that the Fock spaces $\mathcal{F}'_{A,\lambda}$ and $\mathcal{F}'_{A,\lambda'}$ are naturally isomorphic up to a phase. The phase is related to the arbitrariness in filling the Dirac sea between vacuum levels λ, λ'. Such a filling is given corresponds (because of the anticommutation relations) to an exterior product $v_1 \wedge v_2 \wedge \dots v_m$ of a complete orthonormal set of eigenvectors $D_A v_i = \lambda_i v_i$ with $\lambda < \lambda_i < \lambda'$. A rotation of the eigenvector basis gives a multiplication of the exterior product by the determinant of the rotation. Thus there is a well-defined complex line $DET_{\lambda\lambda'}(A)$ for each $A \in U_\lambda \cap U_{\lambda'} = U_{\lambda\lambda'}$ and

$$\mathcal{F}'_{A,\lambda'} = \mathcal{F}'_{A,\lambda} \otimes DET_{\lambda\lambda'}(A) \tag{6.1}$$

over the intersection set. We set $DET_{\lambda'\lambda} = DET^{-1}_{\lambda\lambda'}$ for $\lambda < \lambda'$. Note that from these definitions follows immediately that the line $DET_{\lambda\lambda''}$ can be naturally identified as $DET_{\lambda\lambda'} \otimes DET_{\lambda'\lambda''}$, i.e., the local line bundles $DET_{\lambda\lambda'}$ form a cocycle over the open cover $\{U_\lambda\}$ of $\mathcal{A}$. In order to compensate the dependence on λ in the definition of the Fock spaces we search for a family of complex line bundles DET_λ over the open sets U_λ such that

$$DET_{\lambda'} = DET_{\lambda'\lambda} \otimes DET_\lambda \tag{6.2}$$

over $U_{\lambda\lambda'}$. Obviously, the cocycle property of the of the line bundles $DET_{\lambda\lambda'}$ is a necessary condition for the existence of the family of bundles DET_λ. It is not very hard to prove that this is also a sufficient condition. This follows also form the general theory of bundle gerbes [Mu] since $\mathcal{A}$ is topologically trivial.

We define the tensor product

$$\mathcal{F}_{A,\lambda} = \mathcal{F}'_{A,\lambda} \otimes DET_{A,\lambda}. \tag{6.3}$$

Using (6.1) and (6.2) we observe that the right-hand side is independent from λ and one has a well-defined bundle $\mathcal{F}$ of Fock spaces over all of $\mathcal{A}$.

Next one can ask what is the action of the gauge group in $\mathcal{F}$. The gauge action in U_λ lifts naturally to $\mathcal{F}'$. Thus the only problem is to construct a lift of the action on the base to the total space of DET_λ. Note that the determinant bundle here is a bundle over external fields in *odd dimension,* and therefore one would expect that it is trivial (curvature equal to zero) on the basis of families index theorem. However, it turns out that the relevant determinant bundle actually comes from a determinant bundle in even dimensions. Instead of single vector potentials we must study paths in $\mathcal{A}$, thus the extra dimension. The relevant index theorem is then the APS theorem for even dimensional manifolds with a boundary; physically, the boundary can be interpreted as the union of the space at the present time and in the infinite past, [CaMiMu].

We recall some facts about lifting a group action on the base space X of a complex line bundle to the total space E. Let ω be the curvature 2-form of the line bundle. It is integral in the sense that $\int \omega$ over any cycle is $2\pi\times$ an integer. Let G be a group acting smoothly on X. Then there is an extension $\hat{G}$ which acts on E and covers the G action on X. The fiber of $\hat{G} \to G$ is equal to $Map(X, S^1)$. As a vector space, the Lie algebra of the extension is $\mathbf{g} \oplus Map(X, i\mathbb{R})$. The commutators are defined as

$$[(a,\alpha),(b,\beta)] = ([a,b], \omega(a,b) + \mathcal{L}_a\beta - \mathcal{L}_b\alpha) \tag{6.4}$$

where $a,b \in \mathbf{g}$ and $\alpha,\beta : X \to i\mathbb{R}$. The vector fields generated by the G action on X are denoted by the same symbols as the Lie algebra elements a,b; thus $\omega(a,b)$ is the function on X obtained by evaluating the 2-form ω along the vector fields a,b. The Jacobi identity

$$\omega([a,b],c) + \mathcal{L}_a\omega(b,c) + \text{cyclic permutations} = 0$$

for the Lie algebra extension $\hat{\mathbf{g}}$ follows from $d\omega = 0$.

What we need is a formula for the curvature of the line bundles DET_λ along gauge directions. Not surprisingly, this is given by a reduction from a secondary characteristic class. Recall that in even space-time dimensions the Chern class of the determinant line bundle is obtained by starting from an appropriate characteristic class (the class appearing in the index formula of Dirac operators) in two higher dimensions and then integrating over the space-time manifold; this leaves a closed integral differential form of degree two on the parameter space of the Dirac operators, [AS]. In the odd dimensional case here one starts from the APS index formula on a manifold with a boundary, [APS]. The formula contains two pieces on the right-hand side. The first is an integral of a local differential polynomial (the same as in the case without boundary) and the second is the so-called eta-invariant which contains nonlocal information about the spectrum of the boundary Dirac operator. The essential property of the eta-invariant is that it is gauge invariant. For that reason it does not give a contribution to the curvature of the determinant bundle along gauge orbits. Everything comes from the local differential polynomial; the non- gauge invariant piece of the latter comes from the boundary and is equal to a secondary characteristic class. In simple situations this is just a Chern-Simons form.

Integrating the Chern-Simons form in $2n+3$ dimensions over the $2n+1$ dimensional physical space gives a 2-form along gauge orbits.

For example, when $\dim M = 1$, starting from the Chern-Simons form $\frac{1}{8\pi^2}\mathrm{tr}(AdA + \frac{2}{3}A^3)$ we get

$$\omega_A(X,Y) = \frac{1}{4\pi}\int_{S^1} \mathrm{tr}\, A_\phi [X,Y], \tag{6.5}$$

the curvature at the point A in the directions of infinitesimal gauge transformations X, Y. (Note the normalization factor 2π relating the Chern class to the curvature formula.) This is not quite the central term of an affine Kac-Moody algebra, but it is equivalent to it (in the cohomology with coefficients in $\mathrm{Map}(\mathcal{A}, \mathbb{C})$). In other words, there is a 1-form θ along gauge orbits in $\mathcal{A}$ such that $d\theta = \omega - c$, where

$$c(X,Y) = \frac{i}{2\pi}\int \mathrm{tr} X\partial_\phi Y \tag{6.6}$$

is the central term of the Kac-Moody algebra, considered as a closed constant coefficient 2-form on the gauge orbits. There is a simple explicit expression for θ,

$$\theta_A(X) = \frac{i}{4\pi}\int \mathrm{tr} AX.$$

When $\dim M = 3$ the curvature (or equivalently, the Schwinger term) is obtained from the five dimensional Chern-Simons form

$$CS_5(A) = \frac{i}{24\pi^3}\mathrm{tr}(A(dA)^2 + \frac{3}{2}A^3 dA + \frac{3}{5}A^5).$$

By the same procedure as in the one dimensional case we obtain

$$\omega_A(X,Y) = \frac{i}{4\pi^2}\int \mathrm{tr}\left((AdA + dA\,A + A^3)[X,Y] + XdA\,YA - YdA\,XA\right). \tag{6.7}$$

This differs from the FM cocycle

$$\omega'_A(X,Y) = \frac{i}{24\pi^2}\int \mathrm{tr} A(dXdY - dYdX)$$

by the coboundary of

$$\frac{-i}{24\pi^2}\int \mathrm{tr}(AdA + dA\,A + A^3)X.$$

The Dixmier-Douady class

Let $P\mathcal{F}$ be the bundle of *projective* Fock spaces $\mathcal{F}_A/\mathbb{C}^\times$ over $\mathcal{A}$. Because the action of $\mathcal{G}$ on $\mathcal{A}$ lifts to the total space $\mathcal{F}$ modulo A-dependent phases arising from the Schwinger terms the group $\mathcal{G}$ acts on $P\mathcal{F}$. The action of the subgroup $\mathcal{G}_0$ of based gauge transformations is free and therefore we can define the quotient bundle $P\mathcal{F}/\mathcal{G}_0$ over the manifold $\mathcal{A}/\mathcal{G}_0$. This projective bundle is nontrivial in the sense that there is no Hilbert bundle $\mathcal{H}$ over $\mathcal{A}/\mathcal{G}_0$ whose projectivization would be equal to $P\mathcal{F}/\mathcal{G}_0$, [Se]. The obstruction to constructing the Hilbert bundle is a certain element $\omega_3 \in H^3(\mathcal{A}/\mathcal{G}_0, \mathbb{Z})$, called the Dixmier-Douady class of the bundle, [Br]. The relation of the DD class to quantum field theory was recently clarified by Carey and Murray, [CaMu]; see also [CaMuWa], [M5]. I shall briefly describe the construction of ω_3 below.

We need a locally finite partition of unity on $X = \mathcal{A}/\mathcal{G}_0$ subordinate to the covering by the open sets $V_\lambda = \pi(U_\lambda)$ where $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{G}_0$ is the projection. On a locally compact manifold there exists always a partition of unity subordinate to a given open cover. However, in this case X is not locally compact and we have no proof of the existence of the partition of unity. For that reason we assume that X stands for any finite- dimensional submanifold of $\mathcal{A}/\mathcal{G}_0$ or any other submanifold such that there is a partition of unity $\{f_\lambda\}$ subordinate to the open sets $X \cap V_\lambda$. Let $\theta_{\lambda\lambda'}$ be a representative for the Chern class of the bundle $DET_{\lambda\lambda'}$. Because of the cocycle property of the line bundles we can choose the 2-forms $\theta_{\lambda\lambda'}$ such that

$$\theta_{\lambda\lambda'} + \theta_{\lambda'\lambda''} = \theta_{\lambda\lambda''} \tag{6.8}$$

The forms $\theta_{\lambda\lambda'}$ on $U_{\lambda\lambda'}$ descend to forms on $V_{\lambda\lambda'}$.

We define

$$\theta_\lambda(x) = \sum_{\lambda'} \theta_{\lambda\lambda'}(x) f_{\lambda'}(x) \tag{6.9}$$

at points $x \in V_\lambda$. Now we have

$$\theta_\lambda - \theta_{\lambda'} = \sum_{\lambda''} (\theta_{\lambda\lambda''} - \theta_{\lambda'\lambda''}) f_{\lambda''} = \sum_{\lambda''} \theta_{\lambda\lambda'} f_{\lambda''} = \theta_{\lambda\lambda'} \tag{6.10}$$

by (6.8) and by $\sum f_\lambda(x) = 1$, on the intersection $V_{\lambda\lambda'}$. The forms θ_λ are not closed but on the intersection $V_{\lambda\lambda'}$ we have $d\theta_\lambda = d\theta_{\lambda'}$ since $\theta_{\lambda\lambda'}$ is closed. Thus we may patch together the closed forms $d\theta_\lambda$ to a global closed form ω_3 on X. This is the Dixmier-Douady class of the projective bundle $P\mathcal{F}$.

The above construction is an example of a *bundle gerbe,* introduced by Murray, [Mu] (which in turn is a specialization of the more general theory of gerbes, [Br]). A bundle gerbe is defined as follows. Let $\pi : Y \to X$ be some fibration; in general this is not locally trivial, so Y does not need to be a fiber bundle over X. Let

$$Y^{[2]} = \{(y, y') \in Y \times Y | \pi(y) = \pi(y')\}.$$

A bundle gerbe is a principal $\mathbb{C}^\times$ bundle P over $Y^{[2]}$ with a smooth associative composition map

$$P_{(x,y)} \times P_{(y,z)} \to P_{(x,z)}.$$

The bundle gerbe has also an identity (which is a section of P over the diagonal $Y \subset Y^{[2]}$) and an inverse $P_{(x,y)} \to P_{(y,x)}$, $p \mapsto p^{-1}$.

Example Let $X = \mathcal{A}$ and $Y = \{(A, \lambda) | A \in \mathcal{A},\ \lambda \notin Spec(D_A)\}$. $\pi : Y \to X$ is the natural projection. In this case $Y^{[2]} = \{(A, \lambda, \lambda') | \lambda, \lambda' \notin Spec(D_A)\}$. The bundle P over $Y^{[2]}$ is obtained as the collection of the line bundles $DET_{\lambda\lambda'}$ (with the zero section deleted) over the sets $\{(A, \lambda, \lambda') | A \in U_{\lambda\lambda'}\} \subset Y^{[2]}$. Similarly, a curvature form θ for P is obtained by patching together the local 2-forms $\theta_{\lambda\lambda'}$. The fiber product is given by the natural identification of $DET_{\lambda\lambda'} \otimes DET_{\lambda'\lambda''}$ and $DET_{\lambda\lambda''}$. Since the bundles $DET_{\lambda\lambda'}$ descend to $V_{\lambda\lambda'}$ and the forms $\theta_{\lambda\lambda'}$ are gauge invariant, the whole construction descends to the quationt by $\mathcal{G}_0$ producing a bundle gerbe over $\mathcal{A}/\mathcal{G}_0$.

In the above example the bundle gerbe over $\mathcal{A}$ is trivial (since $\mathcal{A}$) is flat, which means that

$$P = \pi_1{}^*(L) \otimes \pi_2{}^*(L^{-1})$$

for some line bundle L over Y. In our example L is obtained by patching together the local bundles DET_λ. However, the corresponding bundle gerbe over $\mathcal{A}/\mathcal{G}_0$ is nontrivial. The obstruction to trivializing $P/\mathcal{G}_0$ is given by the Dixmier-Douady class $[\omega_3]$.

In general, the DD class is constructed starting from the short exact sequence of de Rham complexes

$$0 \to \Omega^*(X) \xrightarrow{\pi^*} \Omega^*(Y) \xrightarrow{{\pi_1}^* - {\pi_2}^*} \Omega^*(Y^{[2]}) \cap Im({\pi_1}^* - {\pi_2}^*) \to 0.$$

This induces a long exact sequence in cohomology:

$$\cdots \to H^q(X) \xrightarrow{\pi^*} H^q(Y) \xrightarrow{{\pi_1}^* - {\pi_2}^*} {H^q}_\pi(Y^{[2]}) \xrightarrow{\Delta} H^{q+1}(X) \to \dots$$

where H^q_π denotes the image of ${\pi_1}^* - {\pi_2}^*$. The form ω_3 which we constructed above is actually equal to $\Delta(\theta)$. However, the same reservation applies to the use of the exact sequence as we had before in the construction of ω_3 : The construction of the map Δ uses a locally finite open cover, so strictly speaking it is valid only in the case of locally compact manifolds.

7. $\mathcal{U}_{res}$ BUNDLES AND FAMILIES OF FOCK SPACES

Let $H = H_+ \oplus H_-$ be a polarization of a Hilbert space H into a pair of closed infinite-dimensional subspaces. We denote by $\mathcal{U}_{res}$ the restricted unitary group defined by this polarization.

There is an imbedding of $\mathcal{U}_{res}$ in the projective unitary group of the skew symmetric Fock space (determined by the polarization $H = H_+ \oplus H_-$, the 'Dirac sea' construction), [PrSe].

Let Gr be the space of all closed infinite-dimensional subspaces of H with the topology determined by operator norm topology for the associated projections. We may think of Gr as the homogeneous space

$$U(H)/(U(H_+) \times U(H_-)).$$

Here all the groups are contractible (in the operator norm topology) and therefore there is a continuous section $Gr \to U(H)$, that is, for $W \in Gr$ we may choose a $g_W \in U(H)$ which depends continuously on W, such that $W = g_W \cdot H_+$.

The example we shall study below comes from a quantization of a family of Dirac operators D_A parametrized by smooth (static) vector potentials A.

Choose a real number λ such that $D = D_A - \lambda$ is invertible. The set of bounded operators X such that $||X|| < ||1/D||^{-1}$ is an open set V containing 0 and the function $X \mapsto |D+X|^{-1}(D_0 + X)$ is continuous in the operator norm of X; this is seen using the geometric converging geometric series $(D+X)^{-1} = \frac{1}{D} - \frac{1}{D}X\frac{1}{D} + \dots$. Since the operator norm of the interaction A depends continuously on the components A_i of the vector potential (with respect to the infinite family of Sobolev norms on $\mathcal{A}$) we can conclude that $A \mapsto \epsilon_{A,\lambda} = (D_A - \lambda)/|D_A - \lambda|$ is continuous. Thus also the spectral projections $P_\pm(A,\lambda) = \frac{1}{2}(1 \pm \epsilon_{A,\lambda})$ are continuous and $H_+(A,\lambda) = P_+(A,\lambda)H \in Gr$ depends continuously on $A \in U_\lambda$. On the other hand, we know that there is a section $Gr \to U(H)$ and therefore we may choose a continuous function $A \mapsto g_\lambda(A) \in U(H)$ such that $H_+(A,\lambda) = g_\lambda(A) \cdot H_+$. We shall show that these define transition functions, $g_{\lambda\lambda'}(A) = g_\lambda(A)^{-1} g_{\lambda'}(A)$, for a principal $\mathcal{U}_{res}$ bundle P over $\mathcal{A}$. By construction, these satisfy the cocycle property required for transition functions so the only thing which remains is to prove continuity with respect to the topology of $\mathcal{U}_{res}$.

The topology of $\mathcal{U}_{res}$ is defined by the operator norm topology on the diagonal blocks (with respect to the energy polarization $H_+ \oplus H_-$ fixed by the free Dirac operator D_0) and by Hilbert-Schmidt norm topology on the off-diagonal blocks. Denote $P_\pm = P_\pm(A_0, 0)$ and $\epsilon = P_+ - P_-$. We already know that the $g_{\lambda\lambda'}$'s (assume e.g. that $\lambda < \lambda'$) are continuous with respect to the operator norm topology and we need only show that the off-diagonal blocks $[\epsilon, g_{\lambda\lambda'}]$ are continuous in the Hilbert-Schmidt topology. Let us concentrate on the upper right block $K_{+-} = P_+ g_{\lambda\lambda'} P_-$. Multiplying from the left by g_λ and from the right by $g_{\lambda'}^{-1}$ and using the fact that Hilbert-Schmidt operators form an operator ideal with $||gK||_2 \le ||g|| \cdot ||K||_2$ we conclude that K_{+-} is continuous in the Hilbert-Schmidt norm if and only if

$$g_\lambda \pi_+ g_\lambda^{-1} g_{\lambda'} \pi_- g_{\lambda'}^{-1}$$

is a continuous function of A in the Hilbert-Schmidt norm. Now the product of the first three factors in the above expression gives $P_+(A,\lambda)$ whereas the product of the last three factors is $P_-(A,\lambda')$. But $P_+(A,\lambda)P_-(A,\lambda')$ is the spectral projection $P(\lambda,\lambda')$ to the finite-dimensional spectral subspace corresponding to the interval $[\lambda,\lambda']$. On the other hand, the dimension of this subspace is fixed over $U_{\lambda\lambda'}$ and therefore the Hilbert-Schmidt norm of the projection, which is the square root of its rank, is continuous. Furthermore, since $P(\lambda, \lambda')$ is continuous in the operator norm and it has a fixed finite rank it is also continuous in the Hilbert-Schmidt norm.

We denote by Gr_{res} the restricted Grassmannian, defined as the orbit $\mathcal{U}_{res} \cdot H_+$ in Gr. The fiber P_A at $A \in \mathcal{A}$ can be thought of as the set of all unitary operators $T : H \to H$ such that $T^{-1}(H_+(A, \lambda))$ (for any λ) is in Gr_{res}. This is because $g_\lambda(A)$ provides such an operator for any $A \in U_\lambda$ and any two such operators differ only by a right multiplication by an element of $\mathcal{U}_{res}$

Being a principal bundle over a contractible parameter space, $P \to \mathcal{A}$ is trivial.

On any U_λ the function $A \to P_+(A, \lambda)$ is continuous and

$$T_A^{-1} P_+(A, \lambda) T_A \in Gr_{res}.$$

Over Gr_{res} there is a canonical determinant bundle DET_{res}. The action of $\mathcal{U}_{res}$ on Gr_{res} lifts to an action of $\hat{\mathcal{U}}_{res}$ on DET_{res}, [PrSe].

Using these maps $A \to P_+(A, \lambda)$ we can pull back the determinant bundle DET_{res} over Gr_{res} to form local determinant bundles DET_λ over U_λ. This family is the right one for discussing the gerbes over $\mathcal{A}$ and $\mathcal{A}/\mathcal{G}_e$. The reason is that the class of the bundle gerbe is completely determined by the line bundles $DET_{\lambda\lambda'}$ over $U_{\lambda\lambda'}$.

On the restricted Grassmannian we obtain an isomorphism between the fibers $DET_{res}(W)$ and $DET_{res}(W')$, where $W' \subset W$ are points in Gr_{res}; the isomorphism is determined by a choice of basis $\{v_1, \dots, v_n\}$ in $W \cap W'^{\perp}$ as follows. Recalling from [PrSe] that an element in $DET_{res}(W)$ is represented by the so-called admissible basis $\{w_1, w_2, \dots\}$, modulo unitary rotations with determinant equal to one, the isomorphism is simply $\{w_1, w_2, \dots, \} \mapsto \{v_1, \dots, v_n, w_1, w_2, \dots\}$. In particular, we apply this when W, W' are the points obtained by mapping $H_+(A, \lambda)$ and $H_+(A, \lambda')$ to Gr_{res} using T_A. Now the vectors $T_A v_i$ span a basis in the subspace corresponding to the interval $[\lambda, \lambda']$ in the spectrum of D_A and thus they define an element in $DET_{\lambda\lambda'}$ in our earlier construction and the basis can be viewed as an isomorphism between DET_λ and $DET_{\lambda'}$.

Next we consider the trivial bundles $\mathcal{A} \times \mathcal{U}_{res}$ and $\mathcal{A} \times \hat{\mathcal{U}}_{res}$ over $\mathcal{A}$. The gauge group $\mathcal{G}$ acts in the former as follows. Define $\omega(g; A) = T_{g\cdot A}^{-1} g T_A$. This function takes values in $\mathcal{U}_{res}$ and is a 1-cocycle by construction,

$$\omega(gg'; A) = \omega(g; g' \cdot A)\omega(g'; A).$$

Thus the gauge group acts through $g \cdot (A, S) = (g \cdot A, \omega(g; A)S)$ in $\mathcal{A} \times \mathcal{U}_{res}$.

Since ω takes values in $\mathcal{U}_{res}$ the same construction which gives the lifting of the $\mathcal{U}_{res}$ action on Gr_{res} to a $\hat{\mathcal{U}}_{res}$ action on DET_{res} gives also an action of an extension $\hat{\mathcal{G}}$ in $\mathcal{A} \times \hat{\mathcal{U}}_{res}$ and in $\mathcal{A} \times DET_{res}$. The pull-back with respect to the conjugation by T_A's of the latter action defines an action of $\hat{\mathcal{G}}$ on the local determinant bundles DET_λ. Next we observe that the natural action (without center) $v_i \mapsto gv_i$ in the line $DET_{\lambda\lambda'}$ intertwines between the action of the group extension in the lines $DET_\lambda, DET_{\lambda'}$ parametrized by potentials $g \cdot A$ on the gauge orbit. This follows from the corresponding property of the determinant bundle over Gr_{res} (by pushing forward by T_A): An element $\hat{g} \in \hat{\mathcal{U}}_{res}$ acts on $w = \{w_1, w_2, \dots, \} \in DET_{res}(W)$ as $w_i \mapsto \sum_j gw_j q_{ji}$, where the basis rotation q is needed in order to recover a basis in the admissible set. The same element $\hat{g}$ acts then on the basis $w' = w \cup v$ extending the action on w by sending v_i to gv_i.

The intertwining property of the natural action on $DET_{\lambda\lambda'}$ is exactly what was needed in the definition of the action of $\hat{\mathcal{G}}$ in the Fock bundle over $\mathcal{A}$. On the other hand, the obstruction to pushing the Fock bundle over $\mathcal{A}/\mathcal{G}_e$ was precisely the class of the extension $\hat{\mathcal{G}} \to \mathcal{G}$. Thus we have

Theorem. *The obstruction to pushing forward the trivial bundle $\mathcal{A} \times \hat{\mathcal{U}}_{res}$ to a bundle over the quotient $\mathcal{A}/\mathcal{G}_e$, with the action of $\hat{\mathcal{G}}$ coming from the $\mathcal{U}_{res}$ valued cocycle ω, is the Dixmier-Douady class of the Fock bundle.*

It is clear from the above discussion that we may view the Fock bundle over $\mathcal{A}$ as an associated bundle to the principal bundle $\mathcal{A} \times \hat{\mathcal{U}}_{res}$ defined by the representation of $\hat{\mathcal{U}}_{res}$ in the Fock space of free fermions.

Example Let us take a very concrete example for the discussion above. Let $G = SU(2)$ and the physical space $M = S^1$. Now $\mathcal{A}/\mathcal{G}_e$ is simply equal to G since the gauge class of the connection in one dimension is uniquely given by the holonomy around the circle. Because topologically $SU(2)$ is just the unit sphere S^3 any principal bundle over G is described by its transition function on the equator

S^2. In case of a $\mathcal{U}_{res}$ bundle we thus need a map $\phi : S^2 \to \mathcal{U}_{res}$ to fix the bundle and the equivalence class of the bundle is determined by the homotopy class of ϕ. The topology of $\mathcal{U}_{res}$ is known: it consists of connected components labelled by the Fredholm index of $P_+ g P_+$, it is simply connected and so the second homotopy is given by $H^2(\mathcal{U}_{res}, \mathbb{Z}) = \mathbb{Z}$. Thus the equivalence class of a principal $\mathcal{U}_{res}$ bundle over $S^3 \equiv \mathcal{A}/\mathcal{G}_e$ is given by the index of the map ϕ.

The principal $\mathcal{G}_e$ bundle $\mathcal{A} \to \mathcal{A}/\mathcal{G}_e$ is defined by a transition function $\xi : S^2 \to \mathcal{G}_e$. This is determined as follows. Since the total space is contractible, we actually have here a universal $\mathcal{G}_e$ bundle over S^3. Thus the transition function ξ is the generator in $\pi_2(\mathcal{G}_e)$. Such a map can be explicitly constructed. Any point Z on the equator $S^2 \subset S^3$ determines a unique half-circle connecting the antipodes ± 1. We define $g_Z : S^1 \to SU(2)$ by first following the great circle through a fixed reference point Z_0 on the equator, as a smooth function of a parameter $0 \leq x \leq \pi - \delta$ (where δ is a small positive constant), from the point $+1$ to the antipode -1. For parameters $\pi - \delta < x < \pi + \delta$ we let $g_Z(x)$ to be constant, for $\pi + \delta \leq x \leq 2\pi - \delta$ the loop continues from -1 to $+1$ through the point Z on the equator, and finally for $2\pi - \delta \leq x \leq 2\pi$ it is constant. It is easy to see that the set of smooth loops so obtained covers S^3 exactly once and therefore gives a map $g : S^2 \to \mathcal{G}$ of index one.

Any element of $\mathcal{G}$ is represented as an element of $\mathcal{U}_{res}$ through pointwise multiplication on the fermion field in H. Thus by this embedding we get directly the transition function ϕ for the $\mathcal{U}_{res}$ bundle over $\mathcal{A}/\mathcal{G}_e$.

The index of the map ξ can also be checked using the WZWN action,

$$\operatorname{ind} \xi = \frac{1}{24\pi^2} \int_{S^2 \times S^1} \mathrm{TR}(g^{-1} dg)^3$$

and in the fundamental representation of $G = SU(2)$ this gives $\operatorname{ind} \xi = 1$. For chiral fermions on the circle in the fundamental representation of G this is the same as the index of the map $\phi : S^2 \to \mathcal{U}_{res}$. This latter index is evaluated by pulling back the curvature form c on $\mathcal{U}_{res}$ to S^2 and then integrating over S^2. The curvature is defined by the same formula as the canonical central extension of the Lie algebra of $\mathcal{U}_{res}$. Identifying left-invariant vector fields on the group manifold as elements in the Lie algebra we have

$$c(X,Y) = \frac{1}{4}\mathrm{tr}\, \epsilon[\epsilon, X][\epsilon, Y].$$

Note that this curvature on $\mathcal{U}_{res}$ is the generator of $H^2(\mathcal{U}_{res}, \mathbb{Z})$.

The Dixmier-Douady class in our example, as a de Rham class in $H^3(\mathcal{A}/\mathcal{G}_e)$, is simply the normalized volume form on S^3. This is because the third cohomology group of S^3 is one-dimensional and the DD class was constructed starting from the universal bundle $\mathcal{A} \to S^3$.

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