Lorentzian Manifolds

Mattias Blennow, F99
800207-0236
f99-mbl@f.kth.se

5th June 2003

Typeset in LATEX
Abstract

In this essay, the concept of Lorentzian metrics is studied. We begin by giving an introduction to general Riemannian and pseudo-Riemannian metrics in the first section. In the second section we introduce the concept of Lorentzian metric along with some terminology and notation. The third section deals with the Alexandrov topology that may be constructed on a manifold when given a Lorentzian metric. We mainly discuss when this topology is the same as the given manifold topology.

Following the introduction to Lorentzian metrics, we enter into the Čech cohomology theory and use the first Stiefel-Whitney class to determine if a Lorentzian manifold is time and/or space orientable. This is done in the fourth and fifth section.

Finally, in the last section, we briefly look at an application of Lorentzian geometry, namely the theory of general relativity.

To fully understand this essay, the reader should be at least somewhat familiar with differential geometry and cohomology theory.

Throughout this essay, we will mainly use the notation of Beem and Ehrlich [1] for Lorentzian geometry and the notation of Nakahara [2] for general differential geometry.
Contents

1 Riemannian and pseudo-Riemannian metric 4

2 Introducing Lorentzian metrics 7

3 The Alexandrov topology 12

4 Čech cohomology 14
   4.1 The first Stiefel-Whitney class 15

5 Orientability of Lorentzian manifolds 18

6 Lorentzian metrics applied to physics 21
   6.1 Time and space orientability 22
   6.2 The Schwarzschild solution 22
      6.2.1 The perihelion precession 23
      6.2.2 The Schwarzschild black hole 23
   6.3 The Robertson-Walker space-times 24
1 Riemannian and pseudo-Riemannian metric

Given a smooth manifold $\mathcal{M}$ of dimension $m$, we introduce the concept of a metric by the following definition.

**Definition 1.1** A Riemannian metric is a tensor field $g$ of type $(0,2)$ which satisfies the following conditions

\[
g_p(X,Y) = g_p(Y,X) \quad (1.1)
\]

\[
g_p(X,X) \geq 0 \quad , \quad g_p(X,X) = 0 \iff X = 0 \quad (1.2)
\]

for any $p \in \mathcal{M}$ and vectors $X,Y \in T_p\mathcal{M}$.

A pseudo-Riemannian metric satisfies

\[
g_p(X,Y) = 0 \forall Y \iff X = 0 \quad (1.3)
\]

instead of equation (1.2). □

Given a chart $(U_i, \varphi_i)$ of $\mathcal{M}$ with local coordinates $x^\mu$ and some metric $g$ on $\mathcal{M}$, we may express $g$ in local coordinates as

\[
g_p = g_{\mu\nu}(p)d\,x^\mu \otimes d\,x^\nu \quad (1.4)
\]

where $g_{\mu\nu}(p) = g_p\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right)$. Obviously $g_{\mu\nu}(p)$ constitutes the elements of a non-degenerate $m \times m$ matrix for any $p \in \mathcal{M}$.

By construction, the matrix $g_{\mu\nu}(p)$ is diagonalizable with real eigenvalues since it is symmetric. The non-degeneracy also states that all eigenvalues are non-zero. From this follows that if there are $n_+$ positive eigenvalues and $n_-$ negative eigenvalues, then $n_+ + n_- = m$. The combination of numbers $(n_-, n_+)$ is said to be the signature of the metric $g$ (we may also denote this by $(-,\ldots,-,+,\ldots,+)$, with $n_-$ minus signs and $n_+$ plus signs). The signature is clearly independent of the chart on which it is computed.

The signature of a metric is also independent on the point $p$ of the manifold since a change of signature would imply that the metric is degenerate at some point (the eigenvalues may not change discontinuously). Thus, it makes sense to speak about the signature of a metric. By definition, a metric is Riemannian iff it has signature $(0,m)$. 

4
Example 1.2 The manifold $\mathcal{M} = \mathbb{R}^m$ equipped with the usual euclidean metric.

For $\mathbb{R}^m$ we need only the trivial chart given by the euclidean coordinates to cover $\mathbb{R}^m$. In these coordinates the metric tensor is given by

$$g_{\mu\nu}(p) = \delta_{\mu\nu}. \tag{1.5}$$

For any vectors $X, Y \in T_p\mathbb{R}^m = \mathbb{R}^m$, the mapping $g : T_p\mathbb{R}^m \times T_p\mathbb{R}^m \to \mathbb{R}$ is the usual inner product between vectors. □

We now show that an embedding of a manifold $\mathcal{N}$ into a manifold $\mathcal{M}$ equipped with a Riemannian metric $g$ naturally induces a Riemannian metric on $\mathcal{N}$.

Theorem 1.3 If $\mathcal{M}$ and $\mathcal{N}$ are manifolds, $g$ is a Riemannian metric on $\mathcal{M}$ and $f : \mathcal{N} \to \mathcal{M}$ is an embedding, then $h = f^*g$ is a Riemannian metric on $\mathcal{N}$.

Proof: We need to show that equations 1.1-1.2 holds for $h$. First, fix a point $p \in \mathcal{N}$ and take any vectors $X, Y \in T_p\mathcal{N}$, by definition of $h$ and the condition imposed on $g$ by equation (1.1)

$$h_p(X, Y) = g_{f(p)}(f_*X, f_*Y) = g_{f(p)}(f_*Y, f_*X) = h_p(Y, X) \tag{1.6}$$

which states that $h$ also fulfills equation (1.1). That $h_p(X, X) \geq 0$ follows in the same manner.

Since $f$ is an embedding, the map $f_* : T_p\mathcal{N} \to T_{f(p)}\mathcal{M}$ is injective. Thus $f_*X = 0 \iff X = 0$ and it follows that

$$h(X, X) = g(f_*X, f_*X) \geq 0, \quad h(X, X) = 0 \iff X = 0 \tag{1.7}$$

and thus $h$ is a Riemannian metric on $\mathcal{N}$. □

Since any manifold $\mathcal{M}$ of dimension $m$ may be embedded into a higher dimensional Euclidean space, we have the following corollary of Theorem 1.3.

Corollary 1.4 Any manifold $\mathcal{M}$ allows a Riemannian metric. □
In general, Theorem 1.3 does not hold for pseudo-Riemannian metrics as the pullback of \(g\) by the embedding \(f\) is not necessarily non-degenerate and may have different signature at distinct points, in fact there are trivial counter examples (see Example 1.5). However, if the pullback has a globally defined signature, it is a (pseudo-Riemannian) metric.

**Example 1.5** The circle \(S^1\) embedded into \(\mathbb{R}^2\) by the natural embedding \(f(\theta) = (\cos \theta, \sin \theta)\) (with the obvious interpretation of \(\theta\) as a local coordinate). If we equip \(\mathbb{R}^2\) with the pseudo-Riemannian metric \(g = dx \otimes dx - dy \otimes dy\), the pullback \(h = f^*g\) is not a metric on \(S^1\).

Calculating the pullback \(h = f^*g\) we get

\[
h = (\sin^2 \theta - \cos^2 \theta) d\theta \otimes d\theta = - \cos(2\theta) d\theta \otimes d\theta \tag{1.8}
\]

which is degenerate at \(\theta \in \{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\}\) where it also changes signature. □

**Example 1.6** An example of a Riemannian metric induced by an embedding is the induced metric on \(S^2\) from its natural embedding into \(\mathbb{R}^3\).

We equip \(\mathbb{R}^3\) with the Riemannian metric \(g = \delta_{\mu\nu} dx^\mu \otimes dx^\nu\) and embed \(S^2\) into \(\mathbb{R}^3\) by \(f(\theta, \varphi) = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)\). Straightforward calculations give the induced metric tensor on \(S^2\) to be

\[
h = d\theta \otimes d\theta + \sin^2 \theta d\varphi \otimes d\varphi \tag{1.9}
\]

where \(\theta\) and \(\varphi\) are interpreted as the spherical coordinates restricted to some local chart. □
2 Introducing Lorentzian metrics

In this section we introduce the concept of Lorentzian metrics and give a short overview of one of the fields in physics where it is applied, the theory of relativity. We start by giving a definition.

**Definition 2.1 (Lorentzian metric)** Given a manifold $\mathcal{M}$ of dimension $m + 1$, a pseudo-Riemannian metric $g$ is called a Lorentzian metric if it has signature $(1, m)$. The pair $(\mathcal{M}, g)$ is called a Lorentzian manifold.

Lorentzian geometry turns out to be the tool perfectly suited for describing the theory of relativity. As we will notice, Lorentzian geometry and the theory of relativity are so closely related that the terminology of Lorentzian geometry has complete correspondence with the physical interpretations.

**Example 2.2** Lorentzian geometry applied to the theory of general relativity.

General relativity is a theory describing the phenomena known as gravitation. The curvature of the physical space time is given through the Einstein field equations

$$Ric - \frac{1}{2}Rg + \Lambda g = 8\pi T$$  \hspace{1cm} (2.1)

where $Ric$ is the Ricci tensor, $R$ the Ricci scalar, $g$ the metric tensor, $\Lambda$ the cosmological constant and $T$ is the energy-momentum tensor. We have assumed units where $c = G = 1$ (the speed of light and the gravitational constant are set to unity).

A torsion free affine connection is given by the Levi-Civita connection. The trajectories of objects (particles, planets, etc) are given as geodesics. From this, one can predict the perihelion precession of Mercury, the gravitational redshift of light and other physical phenomena. More on this subject can be found in the last section.

We now introduce some terminology. Given a Lorentzian manifold $(\mathcal{M}, g)$ there is a natural decomposition of vectors into three different classes. This decomposition is provided by the Lorentzian metric $g$.

**Definition 2.3** A vector $X \in T_p\mathcal{M}$ is timelike if $g_p(X, X) < 0$, spacelike if $g_p(X, X) > 0$ and null if $g_p(X, X) = 0$. 

7
There is one more important class of vectors, namely the *non-spacelike* vectors. This set of vectors is the union of timelike and null vectors.

In much the same way, we may talk about timelike, spacelike, null and non-spacelike vector fields. A vector field $X$ (a section of $TM$) is said to be timelike if it is timelike at every point, that is

$$g_p(X(p), X(p)) < 0 \quad \forall \ p \in \mathcal{M}. \quad (2.2)$$

Spacelike, null and non-spacelike vector fields are defined in the same manner.

One also defines the timelike and non-spacelike curves. Given a smooth curve $\gamma : I \rightarrow \mathcal{M}$ (where $I$ is some interval of $\mathbb{R}$), we say that it is timelike if the tangent vector $\dot{\gamma}(t)$ is timelike for all $t \in I$. From here we may continue with piecewise smooth curves and say that a piecewise smooth curve is timelike if every piece of the curve is timelike. Non-spacelike curves are defined similarly to timelike curves. Finally, a continuous curve is timelike if for every $t_0 \in I$, there exists a convex normal neighbourhood $U(\gamma(t_0))$ of $\gamma(t_0)$ and an $\varepsilon > 0$ such that $\gamma(t) \in U(\gamma(t_0))$ for all $t \in (t_0 - \varepsilon, t_0 + \varepsilon) = I'$ and for any $t_1, t_2 \in I'$ ($t_1 < t_2$) there exists a smooth timelike curve from $\gamma(t_1)$ to $\gamma(t_2)$ within $U(\gamma(t_0))$.

A convex neighbourhood is a neighbourhood such that any two points of the neighbourhood can be connected by a unique geodesic segment which is entirely within the neighbourhood. For any point $p \in \mathcal{M}$, we may select a basis $\{v_i\}$ of $T_p \mathcal{M} \simeq \mathbb{R}^n$. If $x \in \mathbb{R}^n$ and $|x|$ is sufficiently small, then the map

$$x \mapsto \exp_p \left( \sum_{i=1}^n x_i v_i \right) \quad (2.3)$$

where the exponentiation map is defined as usual\(^1\), is a diffeomorphism from a neighbourhood of the origin in $\mathbb{R}^n$ to a neighbourhood $U(p)$ of $p$ in $\mathcal{M}$. The coordinates $x$ are said to be *normal coordinates* for $U(p)$ based at $p$. A convex normal neighbourhood $U(p)$ is a convex neighbourhood such that for any point $q \in U(p)$, there are normal coordinates for $U(p)$ based at $q$.

We are now ready to define an important concept in the study of Lorentzian manifolds, namely the property of being time orientable.

\(^1\)Let $c_v(t)$ be the unique geodesic in $\mathcal{M}$ such that $c_v(0) = p$ and $\dot{c}_v(0) = v$. Then $\exp_p(v) = c_v(1)$ defines the exponentiation.
Definition 2.4 A Lorentzian manifold \((M, g)\) is said to be time orientable if there exists a global timelike vector field (this field is everywhere non-vanishing by definition). Given such a vector field \(X\), non-spacelike vectors are divided into two classes. A non-spacelike vector \(Y \in T_pM\) is future directed if \(g_p(X(p), Y) < 0\) and past directed if \(g_p(X(p), Y) > 0\). A Lorentzian manifold along with a time orientation is called a space-time.

It is about time to consider an example of a Lorentzian manifold. The simplest example is the Minkowski space-time which plays the same role in Lorentzian geometry as Euclidean space does in Riemannian geometry.

Example 2.5 The \(n+1\)-dimensional Minkowski space-time is given by

\[
(M, g) = \left( \mathbb{R}^{n+1}, -dx^0 \otimes dx^0 + \sum_{k=1}^{n} dx^k \otimes dx^k \right)
\]  

(2.4)

where \(x^\mu\) are the usual Cartesian coordinates on \(\mathbb{R}^{n+1}\). The time orientation (hence “space-time”), is provided by the vector field \(X = \frac{\partial}{\partial x^0}\).

The Minkowski space-time is important for both mathematical and physical reasons. As stated above, it plays the same role in Lorentzian geometry as Euclidean space does in Riemannian geometry. It is also the staging ground of the theory of special relativity describing a universe where there is no gravitation.

Given a space-time \((M, g)\), we may define chronological and causal relations between points in \(M\).

Definition 2.6 Given two points \(p, q \in M\), we say that \(q\) is in the chronological future of \(p\) (or \(p\) is in the chronological past of \(q\)) if there is a timelike future directed (the tangent vector is future directed at each point for smooth curves, etc) curve \(\gamma : [0, 1] \rightarrow M\) such that

\[
\gamma(0) = p, \quad \gamma(1) = q.
\]  

(2.5)

We denote this by \(p \ll q\). In much the same way, we say that \(q\) is in the causal future of \(p\) if there is a non-spacelike curve \(\gamma : [0, 1] \rightarrow M\) fulfilling Equation (2.5). This we denote by \(p \leq q\).
The chronological and causal future (resp. past) of a point \( p \in \mathcal{M} \) are denoted by \( I^+(p) \) and \( J^+(p) \) (resp. \( I^-(p) \) and \( J^-(p) \)). That is
\[
I^+(p) = \{ q \in \mathcal{M} : p \ll q \} \\
I^-(p) = \{ q \in \mathcal{M} : q \ll p \} \\
J^+(p) = \{ q \in \mathcal{M} : p \leq q \} \\
J^-(p) = \{ q \in \mathcal{M} : q \leq p \}.
\]
(2.6)

It is important to note that \( \leq \) is not necessarily a partial order on \( \mathcal{M} \) as we may well have \( p \leq q \) and \( q \leq p \) for distinct points \( p, q \in \mathcal{M} \). In fact, there are space-times for which \( p \ll q \) and \( q \ll p \) for all \( p, q \in \mathcal{M} \). Such a space-time is said to be *totally vicious*. If, however, \( \leq \) is a partial order, the space-time is said to be *causal*. Spacetimes for which \( p \notin I^+(p) \) for all \( p \in \mathcal{M} \) are said to be *chronological*.

In Riemannian geometry, there is the concept of *distance* between points. Given a piecewise smooth curve \( \gamma : I \to \mathcal{M} \), the length of the curve is defined as
\[
L_0(\gamma) = \int_I \sqrt{g(\dot{\gamma}, \dot{\gamma})} \, dt.
\]
(2.7)

The distance between points \( p \) and \( q \) is defined through
\[
d_0(p, q) = \inf_{\gamma \in \Gamma} (L_0(\gamma))
\]
(2.8)

where \( \Gamma = \{ \gamma : [0, 1] \to \mathcal{M} | \gamma(0) = p, \gamma(1) = q \} \).

In Lorentzian geometry, the above definition of distance does not make any sense. As a matter of fact, the expression \( g(\dot{\gamma}(t), \dot{\gamma}(t)) \) is negative if \( \gamma \) is timelike at \( t \). Instead, we use the following definition of Lorentzian distance.

**Definition 2.7** Given a smooth non-spacelike curve \( \gamma : I \to \mathcal{M} \) in a space-time \((\mathcal{M}, g)\), we define the length of the curve as
\[
L(\gamma) = \int_I \sqrt{-g(\dot{\gamma}, \dot{\gamma})} \, dt.
\]
(2.9)

The length of a piecewise smooth curve is defined as the sum of the lengths of the pieces. For a spacelike curve we define \( L(\gamma) = 0 \). We also define the distance between two points \( p, q \in \mathcal{M} \) as
\[
d(p, q) = \sup_{\gamma \in \Gamma} (L(\gamma))
\]
(2.10)
where $\Gamma$ is the set of future directed non-spacelike curves from $p$ to $q$. If not $p \leq q$ then we define $d(p,q) = 0$.

The Lorentzian distance has many properties which are similar to the properties of Riemannian distance. However, some properties are fundamentally different. For instance, in Riemannian geometry, the identity $d_0(p,q) = d_0(q,p)$ always holds which is not the case in Lorentzian geometry.

Since Riemannian distance is defined as an infimum and the Lorentzian as a supremum, there is often a correspondence between the words “minimal” in Riemannian and “maximal” in Lorentzian geometry. For example, in Riemannian geometry we have the triangle inequality

$$d_0(p,q) \leq d_0(p,r) + d_0(r,q)$$

(2.11)

for any points $p$, $q$ and $r$. The corresponding inequality in Lorentzian geometry is the inverse triangle inequality

$$d(p,q) \geq d(p,r) + d(r,q)$$

(2.12)

which holds if $r$ is in the causal future of $p$ and the causal past of $q$.

Even if the sets $I^\pm(p)$ are open (they are the interior of the sets $J^\pm(p)$), the sets $J^\pm(p)$ are not necessarily closed, in fact there are trivial counter examples, see Figure 2.1.
Figure 2.1: The two dimensional Minkowski space with the set $K$ removed is a typical example of a space-time where $J^+(p)$ is not closed for all $p$.

3 The Alexandrov topology

Given a space-time $(\mathcal{M}, g)$, there is a number of ways to construct new topologies on $\mathcal{M}$. The most straightforward of these topologies is the so called Alexandrov topology.

**Definition 3.1 (Alexandrov topology)** The basis of the Alexandrov topology on a space-time $(\mathcal{M}, g)$ are the sets

$$I_{pq} = I^+(p) \cap I^-(q)$$

(3.1)

for all $p, q \in \mathcal{M}$.

Since $I^\pm(p)$ are open sets in the manifold topology, $I_{pq}$ are open sets in the manifold topology as they are finite intersections of open sets. It follows that the original manifold topology is at least as fine as the Alexandrov topology. We may ask ourselves what is required for the Alexandrov topology to be equivalent to the original manifold topology.

**Definition 3.2** In a space-time $(\mathcal{M}, g)$, the open set $U$ is said to be causally convex if no non-spacelike curve intersects $U$ more than once. $(\mathcal{M}, g)$ is said
to be strongly causal at $p \in \mathcal{M}$ if there exists arbitrarily small causally convex neighbourhoods of $p$. A space-time is strongly causal if it is strongly causal at each point.

Being strongly causal turns out to be exactly the condition a space-time must fulfill for the Alexandrov topology to coincide with the given manifold topology.

**Theorem 3.3** For any space-time $(\mathcal{M}, g)$, the Alexandrov topology agrees with the original manifold topology iff $(\mathcal{M}, g)$ is strongly causal.

*Proof:* We first assume that $(\mathcal{M}, g)$ is strongly causal and prove that the Alexandrov topology agrees with the given manifold topology.

Let $U$ be any open set in $\mathcal{M}$. Since $(\mathcal{M}, g)$ is strongly causal, for any point $p \in U$, for which there exists a causally convex set $U_1$ such that $p \in U_1 \subset U$. We may now choose points $p_1, p_2 \in U_1$ such that $p_1 \ll p \ll p_2$ and $I_{p_1 p_2} \subset U_1$ (this is possible since $U_1$ is causally convex). For each point of $U$ we can perform this procedure and assign an open set $I(m)$ in the Alexandrov topology to $m$ such that $I(m) \subset U$. It follows that $U$ is an open set in the Alexandrov topology as it is a union of open sets.

Now suppose that $(\mathcal{M}, g)$ is not strongly causal, that is, there is a point $p \in \mathcal{M}$ such that there exists a convex normal neighbourhood $V(p)$ such that for any neighbourhood $W(p) \subset V(p)$, there exists a non-spacelike curves starting in $W(p)$, leaves $V(p)$ and then returns to $W(p)$. It follows that all neighbourhoods of $p$ contain points outside of $V(p)$ and thus $V(p)$ is not an open set in the Alexandrov topology. Thus, the Alexandrov topology does not agree with the original manifold topology.
4 Čech cohomology

In this section we will deal mainly with the Čech cohomology. In the next section we will use the Čech cohomology to deduce whether a Lorentzian manifold is time (or space) orientable or not. We will be concerned with the group $\mathbb{Z}_2$ and choose to represent it as $\mathbb{Z}_2 = \{1, -1\}$ so that the group operation is the “ordinary” multiplication. We start by defining the Čech cochains.

**Definition 4.1 (Čech cochain)** Let $\mathcal{M}$ be a manifold and $\{U_i\}$ a simple open covering (that is, all $U_i$ and intersections of $U_i$ are contractible) of $\mathcal{M}$. A function $f(i_0, \ldots, i_r) \in \mathbb{Z}_2$, defined on $U_{i_0} \cap U_{i_1} \cap \ldots \cap U_{i_r} \neq \emptyset$ which is totally symmetric under permutation of the arguments, that is

$$f(i_0, \ldots, i_r) = f(i_{P(0)}, \ldots, i_{P(r)}), \quad (4.1)$$

is a Čech $r$-cochain. We denote the multiplicative group of Čech $r$-cochains by $C^r(\mathcal{M}, \mathbb{Z}_2)$.

As usual in cohomology theory, we introduce a coboundary operator $\delta$ which maps $r$-chains to $r+1$-chains with the property that $\delta^2$ maps everything to the trivial element. In this case, the coboundary operator is defined through

$$(\delta f)(i_0, \ldots, i_{r+1}) = \prod_{k=0}^{r+1} f(i_0, \ldots, \hat{i}_k, \ldots, i_{r+1}) \quad (4.2)$$

where $f$ is a Čech $r$-cochain and where the entry under the $\hat{}$ is omitted.

**Example 4.2** For $r = 1$, a cochain assigns the value $\pm 1$ to the overlap of $U_i$ and $U_j$. If $f(i, j)$ is a 1-cochain, its coboundary is given by

$$(\delta f)(i, j, k) = f(i, j)f(i, k)f(j, k) \quad (4.3)$$

The above definition automatically gives $\delta^2 f = 1$ since

$$(\delta^2 f)(i_0, \ldots, i_{r+2}) = \prod_{k \neq m} f(i_0, \ldots, \hat{i}_k, \ldots, \hat{i}_m, \ldots, i_{r+2}), \quad (4.4)$$
\[ f(i_0, \ldots, \hat{i}_k, \ldots, \hat{i}_m, \ldots, i_{r+2}) = f(i_0, \ldots, \hat{i}_m, \ldots, \hat{i}_k, \ldots, i_{r+2}) \text{ and } a^2 = 1 \text{ for any element } a \in \mathbb{Z}_2. \]

This gives rise to the Čech complex

\[ \cdots \xrightarrow{\delta} C^{r+1}(\mathcal{M}, \mathbb{Z}_2) \xrightarrow{\delta} C^r(\mathcal{M}, \mathbb{Z}_2) \xrightarrow{\delta} C^{r-1}(\mathcal{M}, \mathbb{Z}_2) \xrightarrow{\delta} \cdots. \quad (4.5) \]

We may now define the groups of Čech cocycles and Čech coboundaries as

\[
\begin{align*}
Z^r(\mathcal{M}, \mathbb{Z}_2) &= \{ x \in C^r(\mathcal{M}, \mathbb{Z}_2) : \delta x = 1 \} = \ker \delta_r \quad (4.6) \\
B^r(\mathcal{M}, \mathbb{Z}_2) &= \{ x \in C^r(\mathcal{M}, \mathbb{Z}_2) : x = \delta y, \quad y \in C^{r-1}(\mathcal{M}, \mathbb{Z}_2) \} \\
&= \text{Im} \delta_{r-1} \quad (4.7)
\end{align*}
\]

and we use this to define the Čech cohomology groups.

**Definition 4.3 (Čech cohomology groups)** The \( r \)-th Čech cohomology group is defined as

\[
H^r(\mathcal{M}, \mathbb{Z}_2) = \frac{Z^r(\mathcal{M}, \mathbb{Z}_2)}{B^r(\mathcal{M}, \mathbb{Z}_2)} = \frac{\ker \delta_r}{\text{Im} \delta_{r-1}}. \quad (4.8)
\]

Suppose we have two simple coverings \( \{U_i\} \) and \( \{V_i\} \) of \( \mathcal{M} \) that contain the same open sets apart from that two sets in \( \{U_i\} \) are replaced by their union in \( \{V_i\} \). It is easy to see that this does not change the Čech cohomology groups. It follows that the Čech cohomology groups are independent of the specific choice of simple open covering since for two different coverings \( A \) and \( B \). If we start with a simple cover \( C \) with arbitrarily small open sets such that both \( A \) and \( B \) can be formed by replacing elements of \( C \) with unions of elements in \( C \), then it follows that the Čech cohomology groups are the same for both \( A \) and \( B \). It follows that the Čech cohomology groups are well defined for any manifold.

### 4.1 The first Stiefel-Whitney class

Let \( \mathcal{E} \) be a vector bundle over the manifold \( \mathcal{M} \) and let \( \mathcal{E} \) have some inner product (which is, in general, dependent on the point \( p \in \mathcal{M} \)) on the fibre. The \( r \)-th Stiefel-Whitney class \( w_r \) assigns an element \( (w_r(\mathcal{E})) \) of \( H^r(\mathcal{M}, \mathbb{Z}_2) \) to the vector bundle \( \mathcal{E} \). Here we consider the first Stiefel-Whitney class.
If we have some simple covering \( \{ U_i \} \) of the base manifold \( \mathcal{M} \), an element of \( \mathcal{E} \) may be uniquely defined by an element of \( U_i \times \mathcal{F} \), where \( \mathcal{F} \) is the fibre. Also, given an element \( x \in \mathcal{E} \) such that the projection of \( x \) on the base manifold is in \( U_i \), there is a unique element of \( U_i \times \mathcal{F} \) associated to \( x \). On the intersection \( U_i \cap U_j \), any element \( x \in \mathcal{E} \) may be represented by a unique element \( (p, f_i) \in U_i \times \mathcal{F} \) or a unique element \( (p, f_j) \in U_j \times \mathcal{F} \). Clearly, the elements of \( f_i \) and \( f_j \) need not be the same. The transition functions \( t_{ij} : U_i \cap U_j \to \mathcal{G} \) (where \( \mathcal{G} \) is the structure group of the vector bundle), relate \( f_i \) to \( f_j \) by \( f_j = t_{ji}f_i \).

The transition functions must obey certain consistency conditions, namely

\[
\begin{align*}
    t_{ij}(p) &= t_{ji}(p)^{-1} \quad (4.9) \\
    t_{ij}(p)t_{jk}(p)t_{ki}(p) &= 1 \quad (p \in U_i \cap U_j \cap U_k) \quad (4.10)
\end{align*}
\]

Conversely, given a base manifold \( \mathcal{M} \), a simple open covering \( \{ U_i \} \), a vector space \( \mathcal{F} \), a group \( \mathcal{G} \) acting on \( \mathcal{F} \) and transition functions \( t_{ij} : U_i \cap U_j \to \mathcal{G} \), a vector bundle is uniquely defined by taking the union of the sets \( V_i = U_i \times \mathcal{F} \) and identifying \( (p, f_i) \in V_i \) with \( (p, t_{ji}f_i) \in V_j \) for \( p \in U_i \cap U_j \).

Since \( \mathcal{E} \) is equipped with an inner product on the fibre, it makes sense to speak about normalized vectors and the structure group of \( \mathcal{E} \) is \( \mathcal{G} = O(n) \) where \( n \) is the dimension of the fibre.

**Definition 4.4 (The first Stiefel-Whitney class)** Given a vector bundle \( \mathcal{E} \) as above and denoting the transition function from \( U_j \) to \( U_i \) by \( t_{ij} \in \mathcal{G} \), we define a Čech 1-cochain \( f \) by

\[
f(i,j) = \det t_{ij}. \quad (4.11)
\]

The first Stiefel-Whitney class \( w_1 \) is defined by

\[
w_1(\mathcal{E}) = [f] \in H^1(\mathcal{M}, \mathbb{Z}_2). \quad (4.12)
\]

We need to check that this definition is consistent, namely that \([f]\) really is an element of the first Čech cohomology group (that is, \( f \) is an element of the first Čech cocycle group), and that the element does not depend on the
particular choice of the transition functions $t_{ij}$. This is just a simple matter of computation as

$$ (\delta f)(i, j, k) = \det t_{ij} \det t_{ik} \det t_{jk} = \det(t_{ij}t_{jk}t_{ki}) = \det(1) = 1 $$

by the consistency condition imposed on the transition functions and elementary properties of the determinant. That $f(i, k) \in \mathbb{Z}_2$ is obvious as $t_{ij} \in O(n) \implies \det t_{ij} = \pm 1$.

As for the element $[f]$ to be independent of the particular transition functions, the most general change of transition function is given by $t'_{ij} = h_i t_{ij} h_j^{-1}$ (where $h_i : U_i \rightarrow G$ corresponds to a change of fibre basis in $U_i$) which leads to

$$ f'(i, j) = \det t'_{ij} = \det h_i \det h_j \det t_{ij} = (\delta f_0)(i, j)f(i, j) $$

where $f_0(i) = \det h_i$ is a Čech 0-cochain. Since $f$ and $f'$ differ by the coboundary of $f_0$, they represent the same cohomology class and the first Stiefel-Whitney class is well defined.

As we will see in the next section, the first Stiefel-Whitney class can be used to determine if a vector bundle $\mathcal{E}$ is orientable or not.
5 Orientability of Lorentzian manifolds

It is now time to determine when a Lorentzian manifold is time and space orientable.

Let \((M, g)\) be a Lorentzian manifold of dimension \(m+1\). Since the metric \(g\) is symmetric, it is diagonalizable with real eigenvalues and real eigenvectors at each point \(p \in M\). Thus, at a given point \(p\) there are \(m+1\)-eigenvectors \(e_i(p)\) of the metric tensor such that \(g(e_i, e_j) = \lambda(i)\delta_{ij}\), where \(\lambda(i) = -1\) for \(i = 0\) and \(\lambda(i) = 1\) otherwise. The set \(\{e_i(p)\}\) form a basis for \(T_pM\). Any vector \(X \in T_pM\) is naturally split into two parts

\[
X = X^- + X^+ = X_0^\mu e_0^\mu + \sum_{i=1}^m X_i^\mu e_i^\mu.
\]

Thus, the tangent space \(T_pM\) is split into two subspaces, the subspace spanned by \(\{e_0\}\) and the subspace spanned by \(\{e_i: 1 \leq i \leq m\}\). We denote these subspaces by \(T^-_pM\) and \(T^+_pM\) respectively.

Making the above splitting at every point \(p \in M\), any vector field \(X \in TM\) splits as

\[
X(p) = X^-(p) + X^+(p)
\]

where \(X^\pm\) are vector fields with the property that \(X^\pm(p) \in T^\pm_pM\). This gives a splitting of the tangent bundle \(TM\) into a real line bundle \(T^-M\) and an \(m\)-dimensional vector bundle \(T^+M\). A natural inner product on both these bundles is induced through the metric tensor \(g\).

**Definition 5.1** A vector bundle \(E\) of fibre dimension \(n\) with some inner product on the fibre is orientable if it is possible to choose all transition functions \(t_{ij}\) such that \(t_{ij} \in SO(n)\). A manifold \(M\) is orientable if its tangent bundle is orientable. □

The above definition implies that there is some global notion of positive orientation for a fibre basis. In fact, we could skip the requirement for an inner product on the fibre and demand that the transition functions are in \(GL^+(n, \mathbb{R})\) instead of \(SO(n)\), how this generalization affects the reasoning below should be quite clear.

From the splitting of the tangent bundle \(TM\) of a Lorentzian manifold \((M, g)\) into \(TM = T^-M \oplus T^+M\) as well as Definitions 2.4 and 5.1, it should be
clear that \((\mathcal{M}, g)\) is time orientable iff the real line bundle \(T^-\mathcal{M}\) is orientable (there exists a global timelike vector field iff there is a global non-zero section of \(T^-\mathcal{M}\)). Analogously, it also makes sense to define \((\mathcal{M}, g)\) to be space orientable if the vector bundle \(T^+\mathcal{M}\) is orientable.

**Theorem 5.2** A vector bundle \(\mathcal{E}\) with some metric is orientable iff the first Stiefel-Whitney class of \(\mathcal{E}\) is trivial.

**Proof:** Suppose that the first Stiefel-Whitney is class trivial, that is \(f = \delta f_0\) for some Čech 0-cochain \(f_0\). We may always perform a local change of fibre basis for each \(U_i\) by applying \(h_i : U_i \rightarrow O(n)\). It is, of course, possible to choose \(h_i\) such that \(\det h_i = f_0(i)\). The new transition functions \(t'_{ij}\) are given by \(t'_{ij} = h_i t_{ij} h_j^{-1}\) leading to

\[
\det t'_{ij} = \det h_i \det h_j f(i, j) = f_0(i)^2 f_0(j)^2 = 1. \tag{5.3}
\]

It follows that \(t'_{ij} \in SO(n)\).

Now suppose \(\mathcal{E}\) is orientable. By definition of orientability, we may choose the transition functions \(t_{ij}\) such that \(t_{ij} \in SO(n)\) where \(n\) is the fibre dimension. It follows that

\[
f(i, j) = \det t_{ij} = 1 \tag{5.4}
\]

and thus \(w_1(\mathcal{E}) = [f]\) is trivial. \(\square\)

**Example 5.3** The cylinder \(\mathbb{E} = S^1 \times \mathbb{R}\) is orientable while the infintly broad (not needed but comparison to \(\mathbb{E}\) more obvious) Möbius strip \(\mathcal{M}\) is not.

Both \(\mathbb{E}\) and \(\mathcal{M}\) can be simply covered by three open sets \(V_i = U_i \times \mathbb{R}\) where \(U_i\) are open sets of \(S^1\). We may choose \(U_i = \left(\frac{2\pi i}{3}, \frac{2\pi (i+1)}{3}\right)\), \(i \in \{0, 1, 2\}\) with the obvious interpretation of \(\theta \in U_i\) as the polar angle. As a metric on the tangent bundle, we may choose \(g = d\theta \otimes d\theta + dx \otimes dx\).

The transition functions on \(T\mathbb{E}\) may all be taken to be the identity matrix and by the calculations in the proof of Theorem 5.2 and the first Stiefel-Whitney class is trivial. When considering the tangent bundle \(T\mathcal{M}\), it is not possible to choose all transition functions to be in \(SO(2)\) and we must have an odd number of transition functions in \(O(2) \setminus SO(2)\). Let us assume that \(t_{20} \notin SO(2)\) and \(t_{01}, t_{12} \in SO(2)\). This gives

\[
f(i, j) = \begin{cases} 
-1 & \{i, j\} = \{0, 2\} \\
1 & \text{otherwise}
\end{cases} \tag{5.5}
\]
Figure 5.1: The cylinder is orientable while the Möbius strip is not.

However, if \( f \) is a Čech coboundary, then \( f(i, j) = f_0(i)f_0(j) \) for some Čech cochain \( f_0 \). But \( f(0, 1) = f_0(0)f_0(1) = 1 \) and \( f(1, 2) = f_0(1)f_0(2) = 1 \) implies that \( f(0, 2) = f_0(0)f_0(2) = f_0(0)f_0(1)^2f_0(2) = f_0(0)f_0(2) = 1. \) Thus the \( f \) in Equation (5.5) is not a coboundary and \( w_1(TM) \) is not trivial. See Figure 5.1.

According to our previous discussion, Theorem 5.2 has the following corollary:

**Corollary 5.4** A Lorentzian manifold \( (M, g) \) is time orientable iff \( w_1(T^-M) \) is trivial and space orientable iff \( w_1(T^+M) \) is trivial.
6 Lorentzian metrics applied to physics

Like we briefly noted in the beginning of section 2, Lorentzian manifolds are the best tool for describing the general theory of relativity. The curvature and metric tensor are related to the energy momentum tensor through the Einstein field equations (Equation (2.1)) and the Levi-Civita connection provides a torsion free affine connection on the manifold.

In the theory of relativity, we are mainly concerned about four dimensional Lorentzian manifolds for obvious reasons. However, we may often disregard certain dimensions when solving problems (when the solution in that dimension is trivial), and one might always use other dimensions in examples and so on.

As a special case of the theory of general relativity, we get the theory of special relativity which is described by the Minkowski space-time. Most of the predictions that has come out of general relativity are based on particular solutions to the Einstein field equations when the energy momentum tensor has some simple form. In this section we discuss some of those solutions and the predictions they bring.

The solutions we will study all have the cosmological constant $\Lambda$ set to zero. Einstein introduced this constant to prevent his theory from predicting that the universe should either contract or expand. When the expansion of the universe was found empirically, Einstein referred to the cosmological constant as his worst mistake. In recent days it has turned out that the cosmological constant might not have been such a bad mistake, even if its value is not such that it gives the universe a steady state. Never the less, it does have some mathematical interest since the extra term does not prevent the existence of solutions to the field equations.

The trajectories of objects in space-time which are not affected by other forces is given by future directed non-spacelike geodesics (timelike for massive objects and null for massless objects). The proper time between two events for an object (the time elapsed as measured by a “clock” following the object) is the Lorentzian length of the trajectory taking the object from the first event to the second.
6.1 Time and space orientability

In the theory of general relativity, we often assume that the manifold we work with is a space-time. Never the less, we may of course think about the physical consequences if the universe is not time orientable.

The first notable effect is that there is no such thing as a global time direction (by definition). As a result, there is no distinction of future and past, the chronology and causality of events is obviously ill defined and it makes little sense to assume global causality in physical theories (we may still talk about local causality after defining a local time direction).

If the universe is not space orientable, there are also a number of interesting physical consequences. For example, if we send out a right hand glove in the universe and at some later point reunite with it, it might well have become a left hand glove.

Non orientability of time and space also have great consequences on particle physics where we work with time and space reflections as possible symmetries (the time and parity symmetries). For example, it seems like the weak interaction is not symmetric under space reflection and to get a symmetry we need also exchange all particles for anti particles (the charge conjugation symmetry), however, non-orientability of space should indicate that there is no distinction between the different parities.

Finally, if the universe is not time orientable, there is no way of globally defining some of the most important statements of physics, namely the conservation laws like conservation of energy, charge, momentum and so on.

6.2 The Schwarzschild solution

One of the first and most interesting solutions to the field equations is the so called Schwarzschild solution. It is valid outside of a spherically symmetric uncharged and non-rotating mass distribution in space. The Schwarzschild solution is most easily expressed in spherical coordinates for the spatial part outside of \( r = 2M \) (for \( r < 2M \), \( r \) becomes the timelike coordinate) and is given by

\[
g = - \left( 1 - \frac{2M}{r} \right) dt \otimes dt + \left( 1 - \frac{2M}{r} \right)^{-1} dr \otimes dr + r^2 h \tag{6.1}
\]
where \( h = d\theta \otimes d\theta + \sin^2 \theta d\varphi \otimes d\varphi \) is the Riemannian metric on \( S^2 \) induced by the natural embedding in \( \mathbb{R}^3 \) (see Example 1.6).

### 6.2.1 The perihelion precession

In Newtonian mechanics, the gravitational field outside of a spherically symmetric object placed in the origin is given by

\[
g(r) = -\frac{M}{r^3} r
\]

where \( M \) is the mass of the object and \( r = |r| \). This predicts the orbit of any object moving in the potential to be elliptic (this is true even when considering the gravitational effect of the moving object has on the original object, we just have to consider the reduced mass). The perihelion is the point of the orbit which has the least value of \( r \). According to Newtonian mechanics, this point does not change.

Even before the theory of general relativity and the Schwarzschild solution, a perihelion precession of the planet Mercury was observed. That is, the perihelion preceeding around the sun for each Mercury year.

Using the Schwarzschild solution to describe the gravitational effect of the sun, the perihelion precession of Mercury is predicted to be about \( 5.02 \cdot 10^{-7} \) radians per orbit. Taking into account the effect of other planets, this is in remarkably good correspondence to the observed effect. Of course, there is a perihelion precession for the other planets as well. However, this effect is so small that it is barely noticable.

### 6.2.2 The Schwarzschild black hole

One of the most interesting phenomena in physics is the concept of a black hole, a region of space from which not even light might escape. One assumes that all the mass of a spherically symmetric object is gathered within the Schwarzschild radius \( r = 2M \). The metric tensor is now apparently singular at the Schwarzschild radius. However, this apparent singularity is simply due to a poor choice of coordinates and can be removed. This was done by Kruskal...
and Szekeres by a coordinate transformation from $t$ and $r$ to $u$ and $v$ through

\[
uv = (2M - r) \exp \left( \frac{r - 2M}{2M} \right)
\]
\[
t = 2M \ln(-v/u).
\]

(6.3)

(6.4)

In these coordinates, which are valid for $uv < 2M$, the Kruskal-Szekeres metric becomes

\[
g = -\frac{8M^2}{r} \exp \left( \frac{r - 2M}{2M} \right) (du \otimes dv + dv \otimes du) + r^2 h.
\]

(6.5)

The future directed null geodesics of this metric restricted to the $uv$-plane have one of the coordinates increasing and the other coordinate constant. The space is also transformed into four distinct regions which are the quadrants of the $uv$-plane. The first quadrant corresponds to the Schwarzschild black hole, the region with $r < 2M$ and the second quadrant corresponds to $r > 2M$. The fourth quadrant is in some sense a copy of the second and corresponds to $r > 2M$ for a slightly different choice of the coordinate transformation to the Schwarzschild time $t$. The third quadrant is called a “white hole” and has no correspondence in the Schwarzschild solution.

It is not possible to find a future directed non-spacelike geodesic from the black hole region to any of the other regions.

6.3 The Robertson-Walker space-times

Robertson-Walker space-times are space-times formed by Lorentzian warped products between open intervals of the real line and isotropic Riemannian manifolds. That is:

**Definition 6.1** A Robertson-Walker space-time is a space-time $(\mathcal{M}_0 \times_f \mathcal{H}, g)$ such that

\[
(\mathcal{M}_0 \times_f \mathcal{H}, g) = (\mathcal{M}_0 \times \mathcal{H}, -dt \otimes dt + f(t)h)
\]

(6.6)

where $\mathcal{M}_0$ is an interval of the real line, $f : \mathcal{M}_0 \to \mathbb{R}^+$ and $(\mathcal{H}, h)$ is an isotropic Riemannian manifold.
In four space-time dimensions, the Robertson-Walker metric becomes

\[-dt \otimes dt + S(t)^2 \left( \frac{dr \otimes dr}{1 - kr^2} + r^2 d\Omega^2 \right)\]  

(6.7)

where \( S = \sqrt{f} \) and \( d\Omega^2 \) is the Riemannian metric induced on \( S^2 \) by the natural embedding into \( \mathbb{R}^3 \). \( k \) is a constant which by scaling might be set to \( \pm 1 \) or 0.

In physics, this is a solution to an isotropic universe. From this solution, one may deduce that an expanding universe indicates that the universe has been expanding for all earlier values of the global time \( t \). The different values of \( k \) correspond to a universe dense enough to contract, a universe which is not dense enough to contract but where the expansion rate tends to zero and finally a universe where the expansion rate does not tend to zero and the expansion goes on forever.

There are, of course, many other interesting aspects and predictions of general relativity such as the gravitational and cosmological redshifts of light. However, to keep this section brief we leave out these effects which can be studied in practically any book on the subject of general relativity.
References


