Väteatomen

Löstes av Schrödinger 1926.
Visar att Schrödingerekvationen fungerar!

SE i 3D: $-\frac{\hbar^2}{2m}\nabla^2\Psi(\bar{r},t)+V(\bar{r})\Psi(\bar{r},t)=i\hbar\frac{\partial}{\partial t}\Psi(\bar{r},t)$

Sannolikhetstolkning:

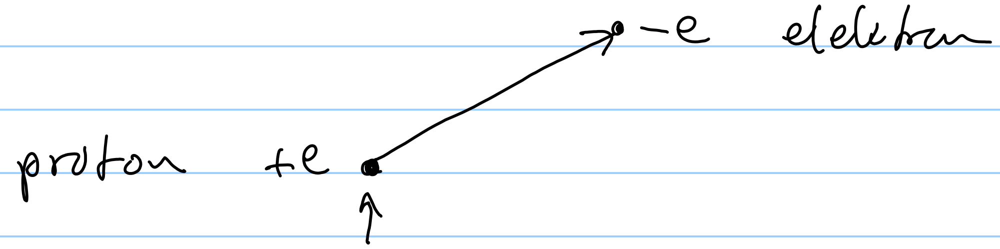
$|\Psi(\bar{r},t)|^2 d^3r$ = sannolikhet att hitta partikeln i volymselementet d^3r i $\bar{r}$ vid tiden t.

Normering: $\int |\Psi(\bar{r},t)|^2 d^3r = 1$

Stationära tillstånd: $\Psi(\bar{r},t)=\psi(\bar{r})e^{-iEt/\hbar}$

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\bar{r})+V(\bar{r})\psi(\bar{r})=E\psi(\bar{r})$$

Väteatomen



antag fix position hos kärnan eftersom $m_p \approx 2000\, m_e$

$$V(\bar{r}) = -\frac{1}{4\pi\epsilon_0}\frac{e^2}{r} \quad \text{Coulomb potential}$$

$$\frac{1}{4\pi\epsilon_0} \simeq 9\cdot 10^9\ Nm^2C^{-2}$$

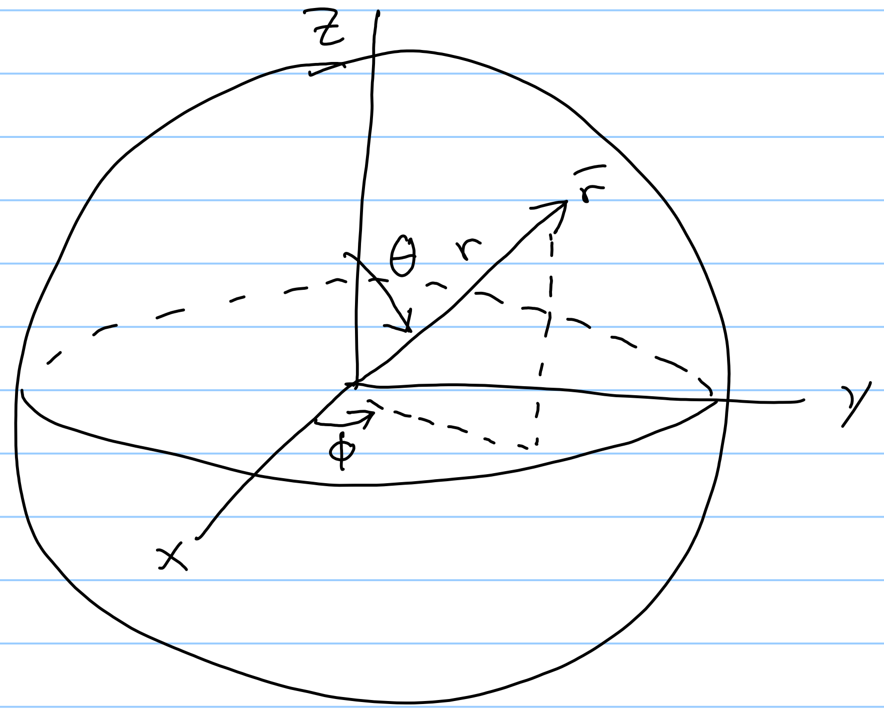
Table with the del operator in cylindrical and spherical coordinates

Operation	Cartesian coordinates (x,y,z)	Cylindrical coordinates (ρ,φ,z)	Spherical coordinates (r,θ,φ)
Definition of coordinates	$\begin{aligned}\rho &= \sqrt{x^2+y^2}\\ \phi &= \arctan(y/x)\\ z &= z\end{aligned}$	$\begin{aligned}x &= \rho\cos\phi\\ y &= \rho\sin\phi\\ z &= z\end{aligned}$	$\begin{aligned}x &= r\sin\theta\cos\phi\\ y &= r\sin\theta\sin\phi\\ z &= r\cos\theta\end{aligned}$
	$\begin{aligned}r &= \sqrt{x^2+y^2+z^2}\\ \theta &= \arctan\left(\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}\right)\\ \phi &= \arctan(y/x)\end{aligned}$	$\begin{aligned}r &= \sqrt{\rho^2+z^2}\\ \theta &= \arctan(\rho/z)\\ \phi &= \phi\end{aligned}$	$\begin{aligned}\rho &= r\sin(\theta)\\ \phi &= \phi\\ z &= r\cos(\theta)\end{aligned}$
Definition of unit vectors	$\begin{aligned}\boldsymbol{\hat\rho} &= \tfrac{x}{\rho}\mathbf{\hat x}+\tfrac{y}{\rho}\mathbf{\hat y}\\ \boldsymbol{\hat\phi} &= -\tfrac{y}{\rho}\mathbf{\hat x}+\tfrac{x}{\rho}\mathbf{\hat y}\\ \mathbf{\hat z} &= \mathbf{\hat z}\end{aligned}$	$\begin{aligned}\mathbf{\hat x} &= \cos\phi\boldsymbol{\hat\rho}-\sin\phi\boldsymbol{\hat\phi}\\ \mathbf{\hat y} &= \sin\phi\boldsymbol{\hat\rho}+\cos\phi\boldsymbol{\hat\phi}\\ \mathbf{\hat z} &= \mathbf{\hat z}\end{aligned}$	$\begin{aligned}\mathbf{\hat x} &= \sin\theta\cos\phi\mathbf{\hat r}+\cos\theta\cos\phi\boldsymbol{\hat\theta}-\sin\phi\boldsymbol{\hat\phi}\\ \mathbf{\hat y} &= \sin\theta\sin\phi\mathbf{\hat r}+\cos\theta\sin\phi\boldsymbol{\hat\theta}+\cos\phi\boldsymbol{\hat\phi}\\ \mathbf{\hat z} &= \cos\theta\mathbf{\hat r}-\sin\theta\boldsymbol{\hat\theta}\end{aligned}$
	$\begin{aligned}\mathbf{\hat r} &= \frac{x\mathbf{\hat x}+y\mathbf{\hat y}+z\mathbf{\hat z}}{r}\\ \boldsymbol{\hat\theta} &= \frac{xz\mathbf{\hat x}+yz\mathbf{\hat y}-\rho^2\mathbf{\hat z}}{r\rho}\\ \boldsymbol{\hat\phi} &= \frac{-y\mathbf{\hat x}+x\mathbf{\hat y}}{\rho}\end{aligned}$	$\begin{aligned}\mathbf{\hat r} &= \tfrac{\rho}{r}\boldsymbol{\hat\rho}+\tfrac{z}{r}\mathbf{\hat z}\\ \boldsymbol{\hat\theta} &= \tfrac{z}{r}\boldsymbol{\hat\rho}-\tfrac{\rho}{r}\mathbf{\hat z}\\ \boldsymbol{\hat\phi} &= \boldsymbol{\hat\phi}\end{aligned}$	$\begin{aligned}\boldsymbol{\hat\rho} &= \sin\theta\mathbf{\hat r}+\cos\theta\boldsymbol{\hat\theta}\\ \boldsymbol{\hat\phi} &= \boldsymbol{\hat\phi}\\ \mathbf{\hat z} &= \cos\theta\mathbf{\hat r}-\sin\theta\boldsymbol{\hat\theta}\end{aligned}$
A vector field $\mathbf{A}$	$A_x\mathbf{\hat x}+A_y\mathbf{\hat y}+A_z\mathbf{\hat z}$	$A_\rho\boldsymbol{\hat\rho}+A_\phi\boldsymbol{\hat\phi}+A_z\mathbf{\hat z}$	$A_r\mathbf{\hat r}+A_\theta\boldsymbol{\hat\theta}+A_\phi\boldsymbol{\hat\phi}$
Gradient ∇f	$\frac{\partial f}{\partial x}\mathbf{\hat x}+\frac{\partial f}{\partial y}\mathbf{\hat y}+\frac{\partial f}{\partial z}\mathbf{\hat z}$	$\frac{\partial f}{\partial \rho}\boldsymbol{\hat\rho}+\frac{1}{\rho}\frac{\partial f}{\partial \phi}\boldsymbol{\hat\phi}+\frac{\partial f}{\partial z}\mathbf{\hat z}$	$\frac{\partial f}{\partial r}\mathbf{\hat r}+\frac{1}{r}\frac{\partial f}{\partial \theta}\boldsymbol{\hat\theta}+\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\boldsymbol{\hat\phi}$
Divergence $\nabla\cdot\mathbf{A}$	$\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}$	$\frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial\rho}+\frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi}+\frac{\partial A_z}{\partial z}$	$\frac{1}{r^2}\frac{\partial(r^2A_r)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta)+\frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$
Curl $\nabla\times\mathbf{A}$	$\begin{aligned}&\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)\mathbf{\hat x}\;+\\ &\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)\mathbf{\hat y}\;+\\ &\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)\mathbf{\hat z}\end{aligned}$	$\begin{aligned}&\left(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi}-\frac{\partial A_\phi}{\partial z}\right)\boldsymbol{\hat\rho}\;+\\ &\left(\frac{\partial A_\rho}{\partial z}-\frac{\partial A_z}{\partial\rho}\right)\boldsymbol{\hat\phi}\;+\\ &\frac{1}{\rho}\left(\frac{\partial(\rho A_\phi)}{\partial\rho}-\frac{\partial A_\rho}{\partial\phi}\right)\mathbf{\hat z}\end{aligned}$	$\begin{aligned}&\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(A_\phi\sin\theta)-\frac{\partial A_\theta}{\partial\phi}\right)\mathbf{\hat r}\;+\\ &\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi}-\frac{\partial}{\partial r}(rA_\phi)\right)\boldsymbol{\hat\theta}\;+\\ &\frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta)-\frac{\partial A_r}{\partial\theta}\right)\boldsymbol{\hat\phi}\end{aligned}$
Laplace operator $\Delta f=\nabla^2 f$	$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}$	$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 f}{\partial\phi^2}+\frac{\partial^2 f}{\partial z^2}$	$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$
Vector Laplacian $\Delta\mathbf{A}=\nabla^2\mathbf{A}$	$\Delta A_x\mathbf{\hat x}+\Delta A_y\mathbf{\hat y}+\Delta A_z\mathbf{\hat z}$	$\begin{aligned}&\left(\Delta A_\rho-\frac{A_\rho}{\rho^2}-\frac{2}{\rho^2}\frac{\partial A_\phi}{\partial\phi}\right)\boldsymbol{\hat\rho}\;+\\ &\left(\Delta A_\phi-\frac{A_\phi}{\rho^2}+\frac{2}{\rho^2}\frac{\partial A_\rho}{\partial\phi}\right)\boldsymbol{\hat\phi}\;+\\ &(\Delta A_z)\mathbf{\hat z}\end{aligned}$	$\begin{aligned}&\left(\Delta A_r-\frac{2A_r}{r^2}-\frac{2}{r^2\sin\theta}\frac{\partial(A_\theta\sin\theta)}{\partial\theta}-\frac{2}{r^2\sin\theta}\frac{\partial A_\phi}{\partial\phi}\right)\mathbf{\hat r}\;+\\ &\left(\Delta A_\theta-\frac{A_\theta}{r^2\sin^2\theta}+\frac{2}{r^2}\frac{\partial A_r}{\partial\theta}-\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\phi}{\partial\phi}\right)\boldsymbol{\hat\theta}\;+\\ &\left(\Delta A_\phi-\frac{A_\phi}{r^2\sin^2\theta}+\frac{2}{r^2\sin\theta}\frac{\partial A_r}{\partial\phi}+\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\theta}{\partial\phi}\right)\boldsymbol{\hat\phi}\end{aligned}$
Differential displacement	$d\mathbf{l}=dx\mathbf{\hat x}+dy\mathbf{\hat y}+dz\mathbf{\hat z}$	$d\mathbf{l}=d\rho\boldsymbol{\hat\rho}+\rho d\phi\boldsymbol{\hat\phi}+dz\mathbf{\hat z}$	$d\mathbf{l}=dr\mathbf{\hat r}+rd\theta\boldsymbol{\hat\theta}+r\sin\theta d\phi\boldsymbol{\hat\phi}$
Differential normal area	$\begin{aligned}d\mathbf{S} =\; &dy\,dz\,\mathbf{\hat x}+\\ &dx\,dz\,\mathbf{\hat y}+\\ &dx\,dy\,\mathbf{\hat z}\end{aligned}$	$\begin{aligned}d\mathbf{S} =\; &\rho\,d\phi\,dz\,\boldsymbol{\hat\rho}+\\ &d\rho\,dz\,\boldsymbol{\hat\phi}+\\ &\rho\,d\rho\,d\phi\,\mathbf{\hat z}\end{aligned}$	$\begin{aligned}d\mathbf{S} =\; &r^2\sin\theta\,d\theta\,d\phi\,\mathbf{\hat r}+\\ &r\sin\theta\,dr\,d\phi\,\boldsymbol{\hat\theta}+\\ &r\,dr\,d\theta\,\boldsymbol{\hat\phi}\end{aligned}$
Differential volume	$dv=dx\,dy\,dz$	$dv=\rho\,d\rho\,d\phi\,dz$	$dv=r^2\sin\theta\,dr\,d\theta\,d\phi$

Non-trivial calculation rules:

1. $\operatorname{div}\operatorname{grad} f=\nabla\cdot(\nabla f)=\nabla^2 f=\Delta f$ (Laplacian)
2. $\operatorname{curl}\operatorname{grad} f=\nabla\times(\nabla f)=0$
3. $\operatorname{div}\operatorname{curl}\mathbf{A}=\nabla\cdot(\nabla\times\mathbf{A})=0$
4. $\operatorname{curl}\operatorname{curl}\mathbf{A}=\nabla\times(\nabla\times\mathbf{A})=\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}$ (using Lagrange's formula for the cross product)
5. $\Delta fg=f\Delta g+2\nabla f\cdot\nabla g+g\Delta f$

Sfäriska koordinater



$$\bar{r} = r\left(\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta\right)$$

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \arccos\frac{z}{r} & \text{(polär vinkel el. zenit, colatitud)} \\ \phi = \arctan\frac{y}{x} & \text{(azimuthvinkel el. longitud)} \end{cases}$$

$$SE: \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) +$$

$$+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} + \frac{2m}{\hbar^2}\underbrace{\left(E-V\right)}_{=\frac{e^2}{4\pi\epsilon_0 r}+E}\psi = 0$$

Variabelseparation: $\Psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$

$$\Rightarrow \frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) +$$

$$+ \underbrace{\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}} + \frac{2mr^2\sin^2\theta}{\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0 r} + E\right) = 0$$

måste vara konstant $= -m_\ell^2$

$$\Rightarrow \frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{2mr^2}{\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0 r} + E\right) =$$

$$= \frac{m_\ell^2}{\sin^2\theta} - \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)$$

måste vara konstant $= \ell(\ell+1)$

Ger 3 ordinära d.e.

$$\begin{cases} \dfrac{d^2\Phi}{d\phi^2} + m_\ell^2\Phi = 0 \\[2ex] \dfrac{1}{\sin\theta}\dfrac{d}{d\theta}\left(\sin\theta\dfrac{d\Theta}{d\theta}\right) + \left[\ell(\ell+1) - \dfrac{m_\ell^2}{\sin^2\theta}\right]\Theta = 0 \\[2ex] \dfrac{1}{r^2}\dfrac{d}{dr}\left(r^2\dfrac{dR}{dr}\right) + \left[\dfrac{2m}{\hbar^2}\left(\dfrac{e^2}{4\pi\epsilon_0 r} + E\right) - \dfrac{\ell(\ell+1)}{r^2}\right]R = 0 \end{cases}$$

Fullständig lösning av dessa ekvationer finns i kommande kurser i kvantmekanik. Vi ska här bara göra de enklaste stegen, och i övrigt ge resultaten utan härledning.

Lösning i ϕ-led:

$$\frac{d^2\Phi}{d\phi^2} + m_\ell^2 \Phi = 0 \Rightarrow \Phi(\phi) = e^{im_\ell \phi}$$

ϕ är 2π-periodisk: $\Phi(\phi + 2\pi) = \Phi(\phi) \Rightarrow$

$$e^{im_\ell(\phi+2\pi)} = e^{im_\ell \phi} \Rightarrow e^{i2\pi m_\ell} = 1$$

$$\Rightarrow m_\ell = 0, \pm 1, \pm 2, \ldots, \pm \ell$$

Kan visa: maximala $|m_\ell|$ är $= \ell$

m_ℓ kallas <u>magnetiska kvanttalet</u>

Kan visa: Energinivåerna ges av

$$\boxed{E_n = -\frac{me^4}{32\pi^2 \epsilon_0^2 \hbar^2}\frac{1}{n^2} = -\frac{13{,}6\ \mathrm{eV}}{n^2}}$$

$n = 1, 2, 3, \ldots$ kallas <u>huvudkvanttalet</u>

$\ell = 0, 1, 2, \ldots, n-1$ kallas <u>bankvanttalet</u>

Bankvantisering

Radiella ekvationen :

vill försöka tolka denna term

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\left[\frac{2m}{\hbar^2}\left(\underbrace{\frac{e^2}{4\pi\varepsilon_0 r}+E}_{}-\frac{\hbar^2 l(l+1)}{2mr^2}\right)\right]R=0$$

tolka detta som $=E_{kin}^{radiell}+E_{kin}^{ban}$

eftersom det gäller (klassiskt): $E=E_{kin}^{radiell}+E_{kin}^{ban}-\dfrac{e^2}{4\pi\varepsilon_0 r}$

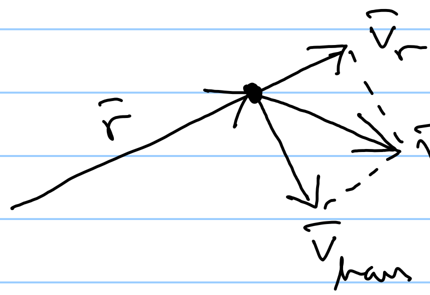
För att den radiella ekvationen ovan endast ska bero på $E_{kin}^{radiell}$ antar vi nu att de två extra energitermerna tar ut varandra

$$\therefore \quad E_{kin}^{ban}=\frac{\hbar^2 l(l+1)}{2mr^2}$$

Vi argumenterar nu för att $E_{kin}^{ban}=\dfrac{L^2}{2mr^2}$

där $\bar{L}=\bar{r}\times\bar{p}$ är <u>rörelsemängdsmomentet</u> :

$$\bar{L}=\bar{r}\times\bar{p}=m\bar{r}\times\bar{v}\Rightarrow L=mrv_{ban}$$



$$\bar{v}=\bar{v}_r+\bar{v}_{ban}\qquad\Rightarrow E_{kin}^{ban}=\frac{1}{2}mv_{ban}^2=\frac{L^2}{2mr^2}$$

Alltså är rörelsemängdsmomentet

$$L^2=l(l+1)\hbar^2\ ,\qquad \boxed{L=\sqrt{l(l+1)}\,\hbar}$$

z-komponenten: $\boxed{L_z=m_l\hbar}$, $m_l=0,\pm1,\dots,\pm l$

Egenfunktioner till rörelsemängdsmomentet

$$\hat{L}^2 Y_\ell^m = \ell(\ell+1)\hbar^2 Y_\ell^m \quad , \quad \ell = 0, 1, 2, \ldots$$

$$\hat{L}_z Y_\ell^m = m\hbar Y_\ell^m \quad , \quad m = -\ell, \ldots, \ell-1, \ell$$

$$\int Y_{\ell'}^{m'*} Y_\ell^m \, d\Omega = \delta_{\ell'\ell} \delta_{m'm}$$

$$d\Omega = \sin\theta \, d\theta \, d\phi$$

Klotfunktioner (spherical harmonics)

$$Y_0^0 = \sqrt{\frac{1}{4\pi}}$$

$$Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos\theta = \sqrt{\frac{3}{4\pi}} \frac{z}{r}$$

$$Y_1^{\pm 1} = \sqrt{\frac{3}{8\pi}} \sin\theta \, e^{\pm i\phi} = \mp \sqrt{\frac{3}{8\pi}} \frac{x \pm iy}{r}$$

$$Y_2^0 = \sqrt{\frac{5}{16\pi}} \left(3\cos^2\theta - 1\right) = \sqrt{\frac{5}{16\pi}} \frac{3z^2 - r^2}{r^2}$$

$$Y_2^{\pm 1} = \sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta \, e^{\pm i\phi} = \mp \sqrt{\frac{15}{8\pi}} \frac{(x \pm iy) z}{r^2}$$

$$Y_2^{\pm 2} = \sqrt{\frac{15}{32\pi}} \sin^2\theta \, e^{\pm 2i\phi} = \sqrt{\frac{15}{32\pi}} \frac{x^2 - y^2 \pm 2ixy}{r^2}$$

osv

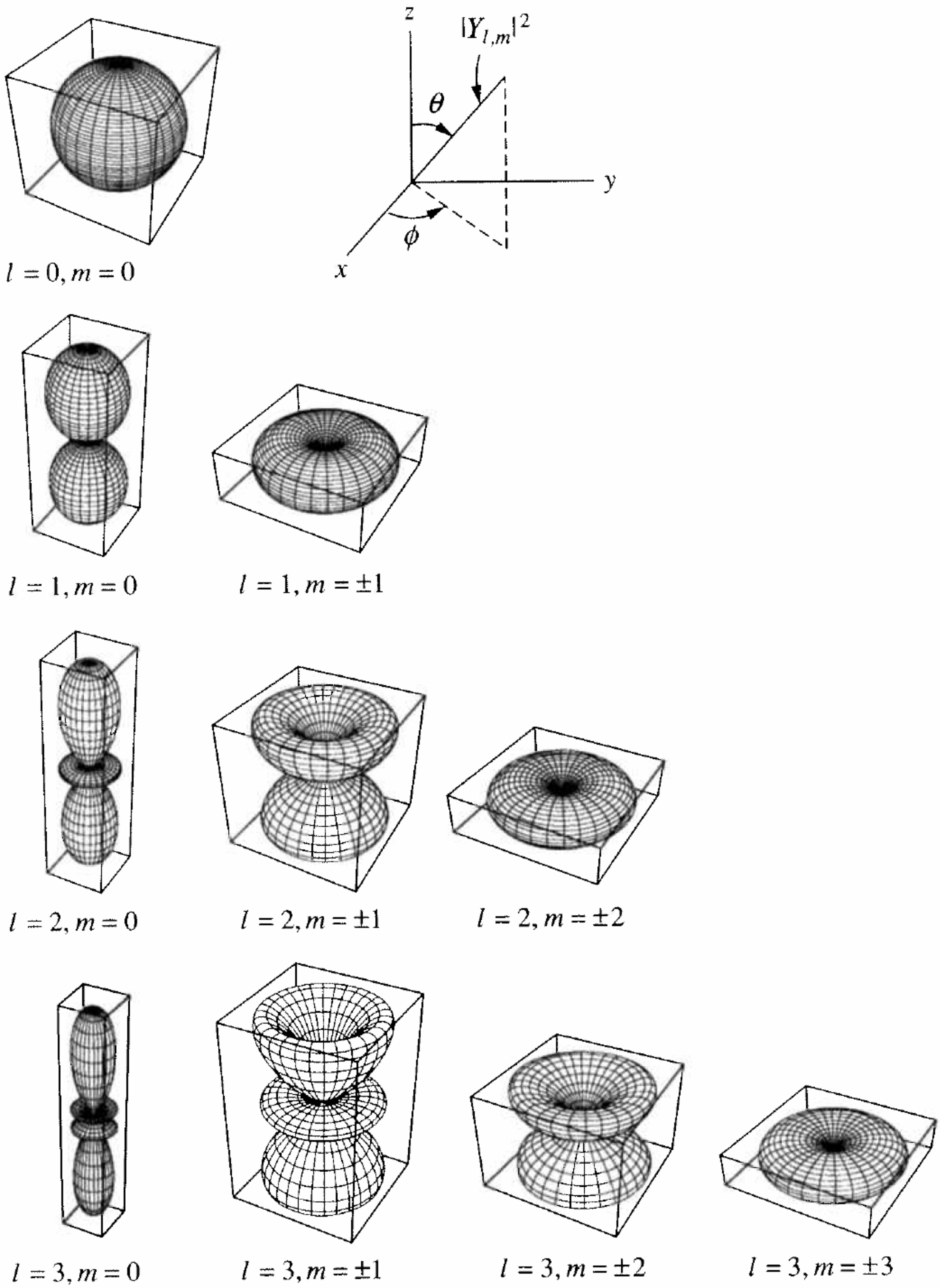


FIGURE 9.11
Plots of $|Y_{l,m}(\theta, \phi)|^2$ for $l = 0, 1, 2$, and 3.

Radiella egenfunktioner

$$R_{nl}(r) = \frac{u_{nl}(r)}{r}$$

Normering: $\int_0^\infty R^*_{n'l'} R_{nl} r^2 dr = \int_0^\infty u^*_{n'l'} u_{nl} dr = \delta_{n'n}\delta_{l'l}$

Bohrradien $a_0 = \frac{4\pi\epsilon_0}{e^2}\frac{\hbar^2}{m_e} = 0.529 \cdot 10^{-10}$ m

Spektroskopisk notation: $l = 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6$
$\qquad s \quad p \quad d \quad f \quad g \quad h \quad i$

1s: $u_{10} = \frac{2}{\sqrt{a_0}}\frac{r}{a_0} e^{-r/a_0}$

2s: $u_{20} = \frac{1}{\sqrt{2a_0}}\left[1 - \frac{1}{2}\frac{r}{a_0}\right]\frac{r}{a_0} e^{-r/2a_0}$

2p: $u_{21} = \frac{1}{2\sqrt{6a_0}}\frac{r^2}{a_0} e^{-r/2a_0}$

3s: $u_{30} = \frac{2}{3\sqrt{3a_0}}\left[1 - \frac{2}{3}\frac{r}{a_0} + \frac{2}{27}\left(\frac{r}{a_0}\right)^2\right]\frac{r}{a_0} e^{-r/3a_0}$

3p: $u_{31} = \frac{4\sqrt{2}}{27\sqrt{3a_0}}\left[1 - \frac{r}{6a_0}\right]\frac{r^2}{a_0} e^{-r/3a_0}$

3d: $u_{32} = \frac{2\sqrt{2}}{81\sqrt{15a_0}}\frac{r^3}{a_0^2} e^{-r/3a_0}$

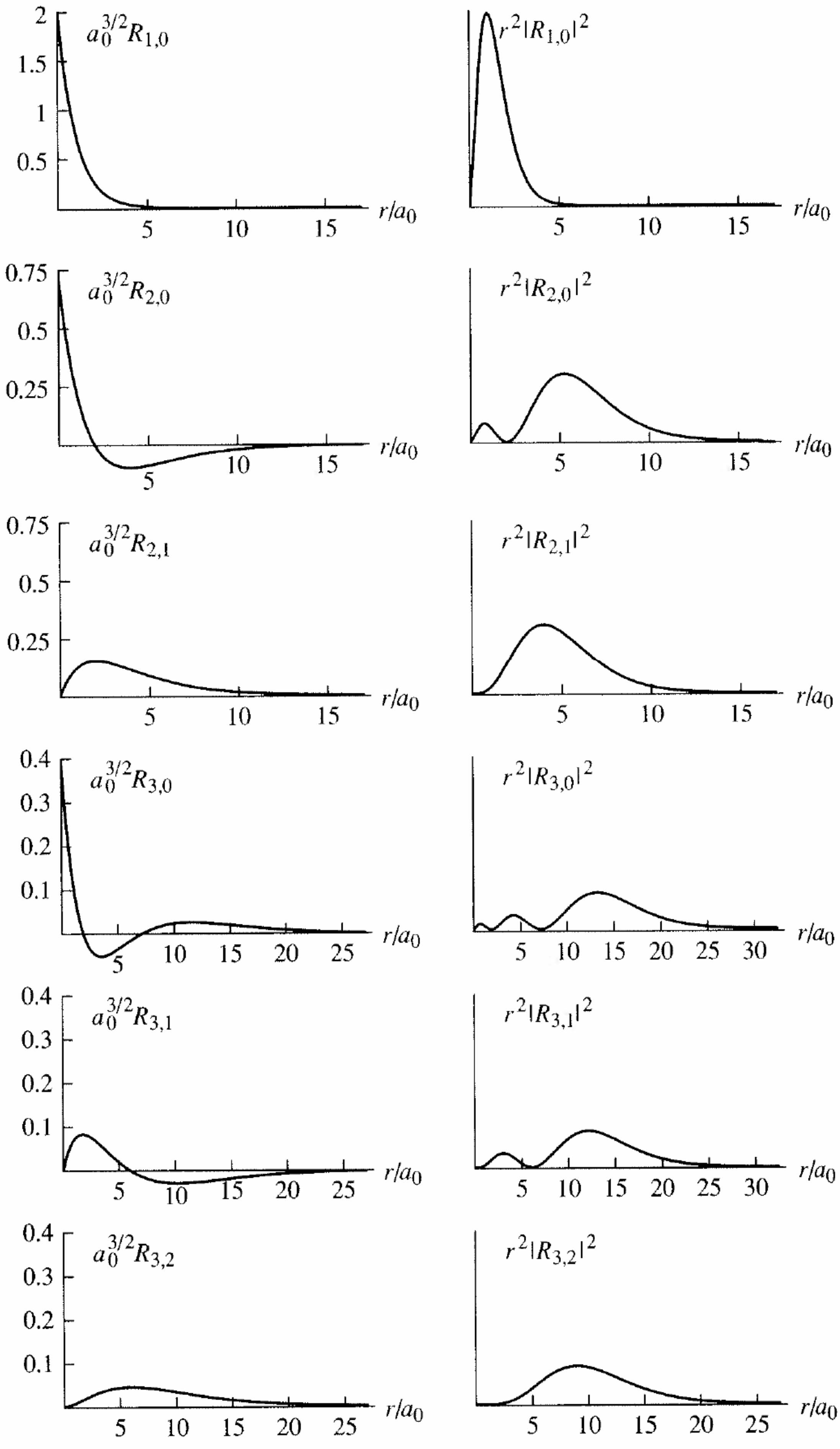


FIGURE 10.5
Plots of the radial wave function $R_{n,l}(r)$ and the radial probability density $r^2|R_{n,l}(r)|^2$ for the wave functions in (10.43), (10.44), and (10.45).

Spektrallinjer

Redan på 1800-talet kände man till att väte emitterar ljus enbart med våglängder

$$\frac{1}{\lambda} = R_H \left(\frac{1}{4} - \frac{1}{n^2} \right), \quad n = 3, 4, 5, \ldots, \quad R_H \cong 10^7 \, m^{-1}$$

Detta är den sk Balmerserien för den synliga delen av vätes spektrum.

Denna egenskap förklaras inom kvantmekaniken:
Om en elektron övergår från $n_i \to n_f$ så strålar ut energin i form av en foton med energi

$$E = E_i - E_f = -\frac{me^4}{2(4\pi\epsilon_0)^2 \hbar^2} \left[\frac{1}{n_i^2} - \frac{1}{n_f^2} \right] = hf = \frac{hc}{\lambda}$$

$$\Rightarrow \frac{1}{\lambda} = \frac{E}{hc} = \underbrace{\frac{me^4}{2(4\pi\epsilon_0)^2 \hbar^2 hc}}_{= 1{,}097 \cdot 10^7 \, m^{-1}} \left[\frac{1}{n_f^2} - \frac{1}{n_i^2} \right]$$

dvs samma som formeln ovan med $n_f = 2$, $n_i = 3, 4, 5, \ldots$ Detta är Balmerserien, synligt ljus.
$n_f = 1$: Lymanserien, UV ljus
$n_f = 3$: Paschenserien, IR ljus

Detta var en triumf för den nya kvantmekaniken.