Relativity

Newtonian mechanics: a world without a speed limit

Choose a reference frame = coordinate system in 3D space $\bar{r} = (x, y, z)$.

Consider a mass point m. If the reference frame is far from any external influences, the particle should move in a straight line at a constant velocity (first law). Such frames are called inertial.

If the point mass feels a force it accelerates as

$$\bar{F} = m\bar{a}$$

(second law).

If a particle acts with a force $\bar{F}_1$ on another particle, then the second particle acts with a force

$$\bar{F}_2 = -\bar{F}_1$$

on the first (third law). This is the law of Action-Reaction, and implies

$$m_1\bar{a}_1 = -m_2\bar{a}_2$$

Hence if we define $m_1 = 1$ then m_2 follows.

New inertial frames can be generated from required symmetry of physical laws:

1) Space is uniform: laws symmetric under translations. The origin can be moved which gives a new inertial frame.

2) Space is isotropic: laws symmetric under rotations. New inertial systems obtained by rotation around some axis.

3) Galilean invariance: laws symmetric under adding a constant velocity.
New inertial frames obtained by a <u>boost</u>: a frame moving at constant velocity to the first.

It is easily verified that 1-3 are fulfilled by Newtons law $\bar{F} = m\bar{a}$. For example, adding a constant velocity does not change the acceleration.

Structure of space and time

To measure distances we use a measuring rod (ruler). Starting from the origin we mark positions in x, y, z directions. We can now measure positions $\bar{r} = (x, y, z)$ in this inertial frame. To measure time we need a clock. Any cyclic mechanical device will do. An accurate clock requires a short period. Accurate clocks using atomic oscillations do this job.

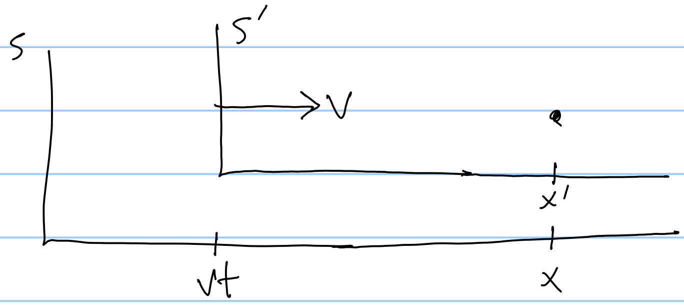
Assume that we place a clock at each distance marker in the frame. We can then measure <u>events</u>: $\vec{r}, t$ when particles pass by etc. The clocks can be syncronized by putting a signal source between them that sends out a signal in all directions. The clocks are set to zero when the signal arrives to both clocks. All clocks are syncronized in a similar way.

The central difference between Newtonian mechanics and relativity is the maximum speed of transmission of information: it is infinite in the Newtonian world and finite $(=c)$ in relativity.

The possibility of unbounded transmission speed of information is related to the possibility according to Newtons laws to accelerate to arbitrarily large velocity: a mass m with charge q gets velocity $v = \frac{qE}{m} t$. This means that we can in principle syncronize two inertial frames moving at constant relative velocity, so that if we have one measuring rod and one clock, we know all measuring rods and clocks in all other frames as well. This corresponds to the laws of <u>absolute space</u> and <u>absolute time</u>. They come from the fact that in Newtonian mechanics there is no speed limit — information can be transmitted instantaneously. Also forces and potentials transmit instantaneously. Relativity is very different as we will see.

Galilei transformation

The transformation between two coordinate systems S, S' in relative motion with constant speed v



$$x = x' + vt \qquad \text{absolute space}$$
$$y = y'$$
$$z = z'$$
$$t = t' \qquad \text{absolute time}$$

The distance x is the position vt of the origin of S' plus the position x'. x' is measured to be the same in both frames S and S' by absolute space.

Velocity addition

If the particle has velocity v_p' in S' in the x'-direction then $x' = v_p' t' = v_p' t$.
Its position in S is $x = vt + x' = (v + v_p')t$.
Hence the velocity in S is

$$v_p = v_p' + v$$

Physics according to Einstein: a world with a speed limit.

Accelerator experiments, not available to Newton, demonstrate that particles can not be accelerated to velocities greater than the universal speed limit, the speed of light:

$$c = 2.99792458 \cdot 10^8 \text{ m/s} \approx 3 \cdot 10^8 \text{ m/s}$$

(The meter is defined so that this is exact in vacuum.)

This immediately shows that Newtonian absolute space and time is wrong!

Einstein's relativity builds on two postulates:

1. The laws of physics are the same in all inertial frames

2. There is a universal speed limit: c

By Galilei velocity addition the speed of light in frame S' from a source in S is

$$v_p = v + c$$

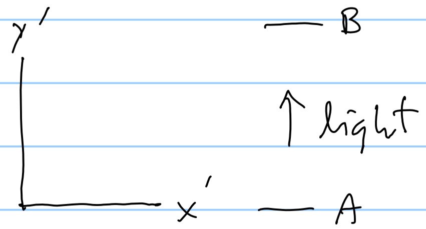
This contradicts experiments, that always get c irrespective of what sends out the light.

In relativity, the speed limit must be the same in S and S' (by postulate 1) and both S and S' observe speed c.

We need to work out the implications of these two postulates. Start by figuring out how rods and clocks are related in S and S', then study how Newton's laws must be modified and how momentum and energy must be defined and related.

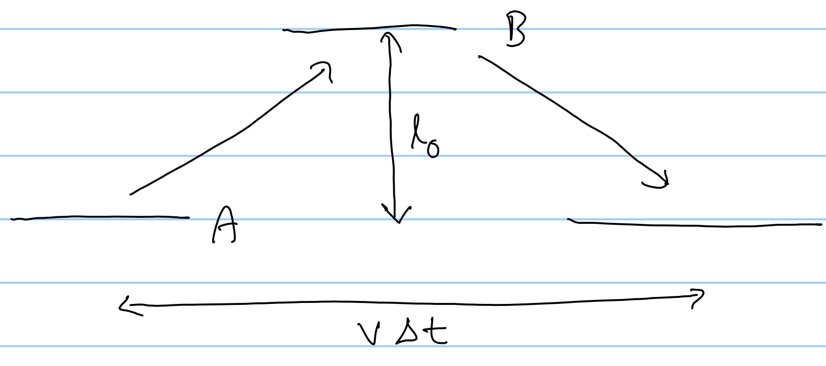
A simple clock and time dilation

Make a basic clock at rest in S'. Take two mirrors with separation l_0 in the y' direction and let light bounce back and forth between them. The period is $\Delta t' = 2l_0/c$ to bounce $A \to B \to A$.



Since the clock is at rest in y' we call $\Delta\tau = 2l_0/c$ the proper time. We can place such clocks along the gridmarks in S' to measure the position and time of events.

Now view the clock's operation from rest at S.



The clock now moves at velocity v and the light ray takes the path $AB \to BA$ in the figure. The distance the mirror A travels between sending and receiving the light is $v\Delta t$. The distance the light travels is

$$\overline{AB} + \overline{BA} = 2\sqrt{l_0^2 + \left(\frac{v\Delta t}{2}\right)^2} = c\,\Delta t$$

Use the speed of light is universal!

$$\frac{4\,l_0^2}{c^2} + \frac{v^2\Delta t^2}{c^2} = \frac{c^2\,\Delta t^2}{c^2}$$

$$\frac{4\,l_0^2}{c^2} = \Delta\tau^2 = \left(1 - \frac{v^2}{c^2}\right)\Delta t^2$$

$$\boxed{\Delta t = \frac{\Delta\tau}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma\,\Delta\tau} > \Delta\tau \ !$$

where $\boxed{\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} > 1}$ is the <u>Lorentz factor</u>.

Compare to the same result in Newtons world: The velocity in S is $v_x = v$, $v_y = c \Rightarrow$ speed $\sqrt{c^2+v^2} \Rightarrow$

$$2\sqrt{l_0^2 + \left(\frac{v\Delta t}{2}\right)^2} = \sqrt{c^2+v^2}\ \Delta t$$

$$\Rightarrow \quad 2\,l_0 = c\,\Delta t \Rightarrow \Delta t = \frac{2\,l_0}{c} = \Delta\tau$$

In the real world of Einstein, there is a universal speed limit, and $\Delta t > \Delta\tau$. The universal speed limit directly produces time dilation!

How big is the difference? If $v = 300\ m/s$ so $v/c = 10^{-6}$ then

$$\Delta t = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\Delta\tau$$

(Calculation help: 1) $\sqrt{1+x} \approx 1 + \frac{x}{2} + \mathcal{O}(x^2)$

Proof: square both sides

$$1+x = \left(1+\frac{x}{2}+\mathcal{O}(x^2)\right)^2 = 1+x+\mathcal{O}(x^2) \qquad \underline{ok}$$

2) $$\frac{1}{1-x} = 1+x+\mathcal{O}(x^2)$$

Proof multiply by $1-x$:

$$1 = (1-x)\left(1+x+\mathcal{O}(x^2)\right) = 1+\mathcal{O}(x^2) \qquad \underline{ok})$$

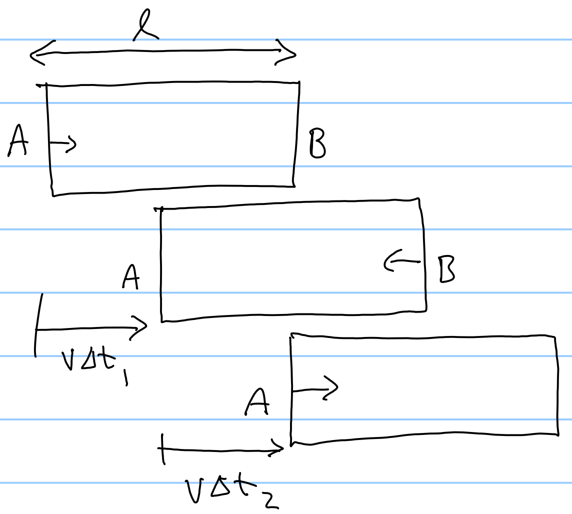
$$\Rightarrow \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{1-\frac{1}{2}\frac{v^2}{c^2}+\mathcal{O}\left(\frac{v^4}{c^4}\right)} = 1+\frac{1}{2}\frac{v^2}{c^2}+\mathcal{O}\left(\frac{v^4}{c^4}\right)$$

$$\therefore \quad \Delta t \approx \Delta\tau\left(1+\frac{1}{2}\frac{v^2}{c^2}\right) \approx \Delta\tau\cdot\left(1+0.5\cdot 10^{-12}\right)$$

This is why we normally never see relativistic effects. In accelerators relativistic effects are common. More later.

Lorentz contraction

Turn the clock in the x' direction, so its length l_0 is parallel to the velocity v of S' relative to S. l_0 is the clocks proper length since it is the length in its rest frame. Since space is isotropic the rotated clock works in the same way as before. In S' we have $\Delta\tau = 2l_0/c$, and in S the proper time is dilated to $\Delta t = \gamma\Delta\tau$. View the light ray in S:



light leaves A

light reaches B after Δt_1

light reaches A after Δt_2

$$\therefore \quad c\Delta t_1 = l + v\Delta t_1 \quad \Rightarrow \quad \Delta t_1 = \frac{l}{c-v}$$

$$c\Delta t_2 = l - v\Delta t_2 \quad \Rightarrow \quad \Delta t_2 = \frac{l}{c+v}$$

$$\Delta t = \Delta t_1 + \Delta t_2 = \frac{l}{c-v} + \frac{l}{c+v} = \frac{l(c+v)+l(c-v)}{c^2-v^2} =$$

$$= \frac{2l/c}{1-\frac{v^2}{c^2}} = \gamma\,\Delta\tau = \frac{2l_0/c}{\sqrt{1-\frac{v^2}{c^2}}}$$

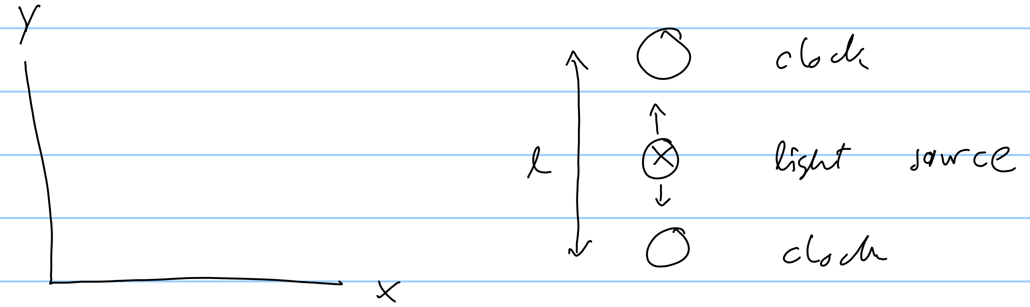
$$\therefore \quad \boxed{l = l_0\sqrt{1-\frac{v^2}{c^2}} = l_0/\gamma} < l_0$$

This was discovered by H.A. Lorentz in the context of electromagnetism about year 1900.
But it was only with Einstein's derivation that the universality of the result was clarified.

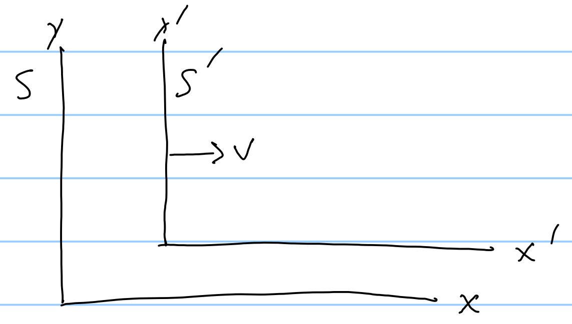
Hence an observer measures the length of a moving rod to be contracted by a factor γ in the direction of motion. This follows from the universal speed limit by the derivation above. But why? No forces are involved, so what does it mean that the rod is contracted? It will clarify when we consider syncronization of spatially separated moving clocks.

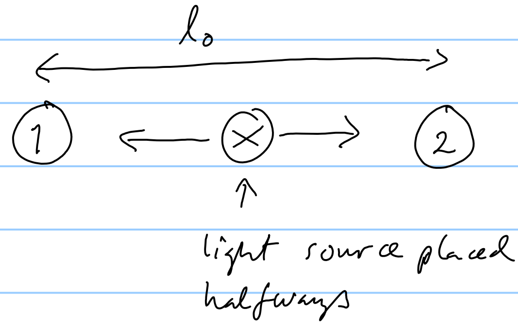
Relative simultaneity

Consider syncronization of clocks.
A clock in the y' direction is easy to syncronize using a light signal in the middle.
This syncronizes clocks in both S' and in S



It gets more interesting to syncronize clocks separated in the x' direction





Let $\delta x' = l' = l_0$ be the clock separation in their rest frame S'.
At $t'=0$ light is emitted and both clocks register $t' = l_0/2c$ when the light arrives.
This is agreed by all observers in all inertial systems.

Seen from S, light reaches 1 before 2 since 1 is moving towards the source and 2 away from the source. The time to reach the clocks are

$$ct_2 = \frac{l}{2} + vt_2 \quad \Rightarrow \quad t_2 = \frac{l/2}{c-v}$$

$$ct_1 = \frac{l}{2} - vt_1 \quad \Rightarrow \quad t_1 = \frac{l/2}{c+v}$$

which uses the universal nature of c.
Hence the clocks are not syncronized in S, or in any other frame in motion wrt S'!
This follows since c is the same in all frames, and violates Galilean velocity addition.

The time difference seen in S is

$$\Delta t = t_2 - t_1 = \frac{\ell/2}{c-v} - \frac{\ell/2}{c+v} = \frac{(c+v)-(c-v)}{c^2-v^2}\frac{\ell}{2}$$

$$= \frac{\ell v/c^2}{1-\frac{v^2}{c^2}} = \gamma \frac{\ell_0 v}{c^2} = \gamma \Delta t'$$

$\ell = \ell_0\sqrt{1-\frac{v^2}{c^2}}$ by time dilation

Hence from S it is found that $\Delta t' = \ell_0 v/c^2$ is the time difference in S' between the time when the light hits the clocks.
But all observers in all frames agree that 1 and 2 have identical readings when the light hits them.

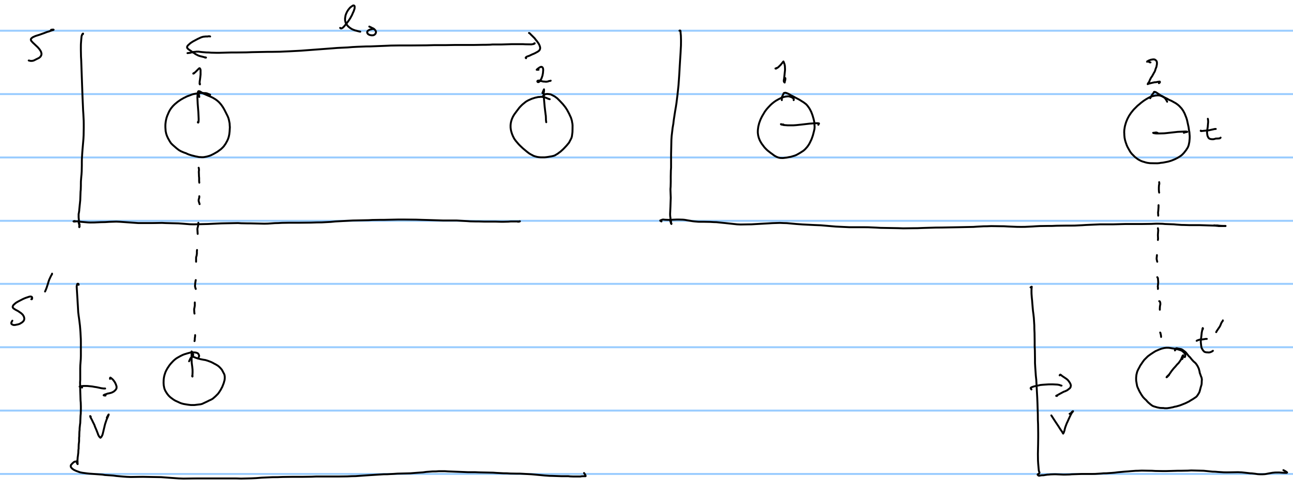
$\boxed{\text{Hence from } S \text{ 1 was set ahead of 2 by } \ell_0 v/c^2}$

Now we have the 3 effects that must be taken together to understand relativity: time dilation, length contraction, and relativity of simultaneity.

Time dilation again

Both the observer in S and in S' finds that the other clock runs slow. How can these observations agree with each other?

Study two syncronized clocks at rest in S and one clock in S' that syncronizes with clock 1 in S when they pass by at $x = x' = 0$, $t = t' = 0$:

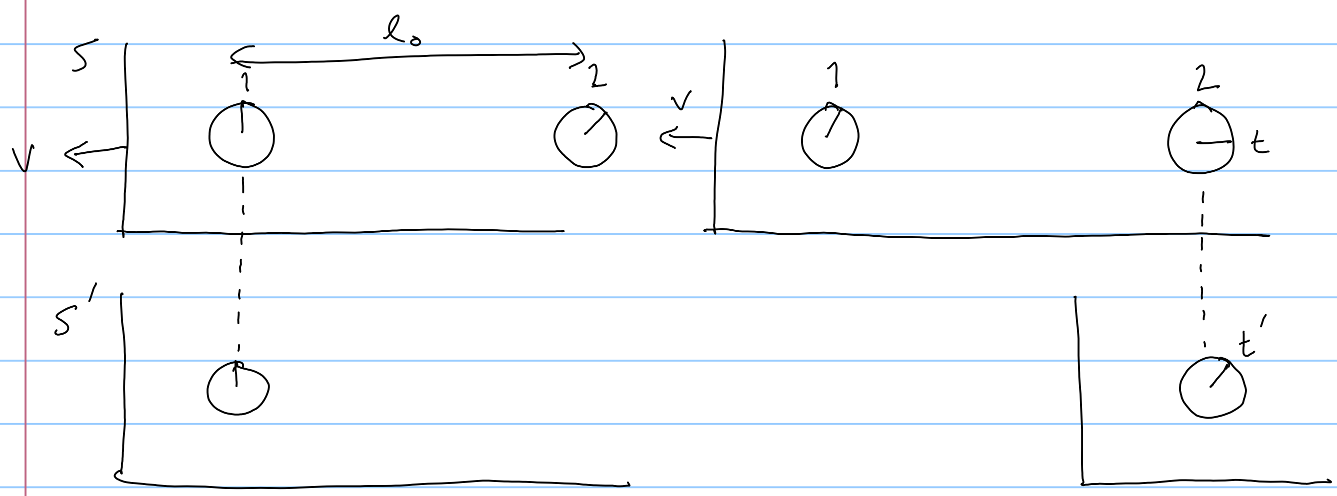


Seen from S:
The clock in S' passes 2 at $t = l_0/V$ and has slowed down to

$$t' = \frac{t}{\gamma} = \frac{l_0/V}{\gamma}$$

The clock in S' is not syncronized with 2.

Seen from S', S moves to the left with velocity v:



S' sees the distance between the clocks contracted to $l' = l_0/\gamma$, so the clock reads $t' = l'/v = l_0/\gamma v$ when it reaches 2.
Hence S and S' agree on this, as they must!
But what does clock 2 read?
At $t = t' = 0$, S' finds 2 to be $l_0 v/c^2$ ahead of 1.
At $t' = l_0/\gamma v$ the clocks in S' and 2 meet, and then the time $t = t'/\gamma$ has passed in S.
So clock 2 reads

$$t = \frac{l_0 v}{c^2} + \frac{l_0}{v}\left(1 - \frac{v^2}{c^2}\right) = \frac{l_0}{v}$$

which agrees with what S sees.

Hence both observers sees the other's clock run slow, and both agree that

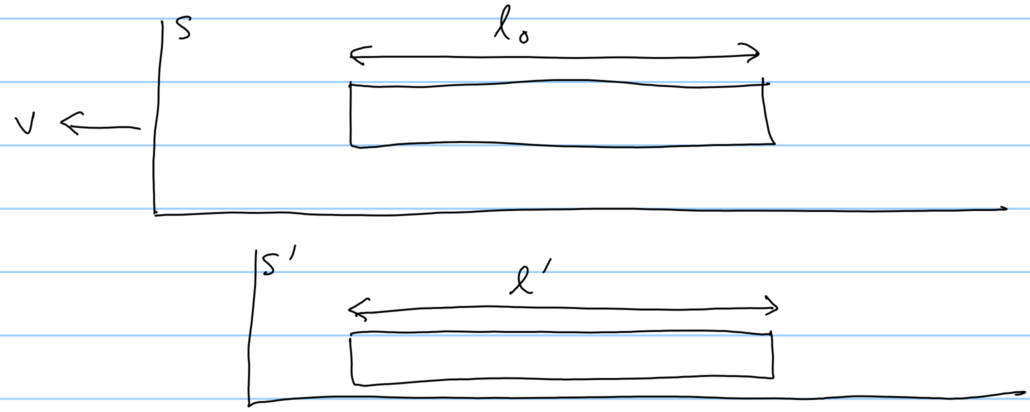
$$t = l_0/v, \qquad t' = l_0/\gamma v$$

when the clocks meet.

Length contraction again

Both S and S' see lengths contract in the other frame. How can they agree?

Consider a rod of length l_0 at rest in S. Let S move to the left wrt S' with velocity v.

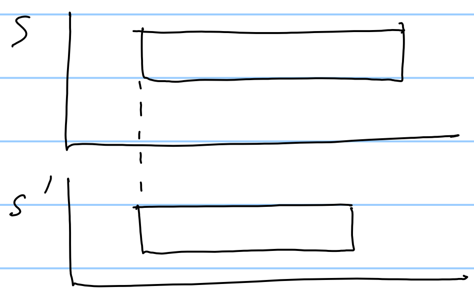


Seen from S' the length is

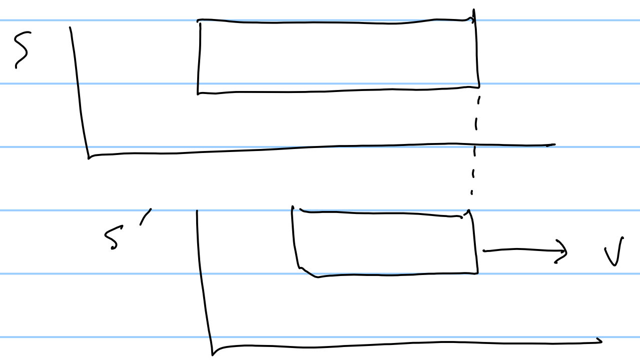
$$l' = f\,l_0$$

where f is a factor that we will construct and understand. The observer in S' measures l' at <u>one</u> instant in his frame.

From rest in S, there is a rod in S' of length l' in S', moving to the right. At $t = t' = 0$ the left ends coincide



By relativity of simultaneity, S finds the clock at the right end in S' to read $t' = -l'v/c^2$. Later the right ends of the rods coincide so that the clock on the end of the rod in S' reads $t' = 0$ since S' makes a <u>single</u> measurement in S'.



The time elapsed is $l'v/c^2$ in S', which dilates to $\gamma l' v/c^2$ in S.
When the observer in S measures the length of the rod in S', it reads $f l' = f^2 l_0$.
Therefore l_0 can be measured as

$$\underbrace{f^2 l_0}_{l' \text{ seen from } S} + \underbrace{\left(\gamma l' v/c^2\right)\cdot v}_{\text{distance the right end moves}} = l_0$$

$$= f\gamma l_0 v^2/c^2$$

Solutions: $f = \frac{1}{\gamma} = \sqrt{1-\frac{v^2}{c^2}}$ ($f = -\gamma$ is unphysical)

(since $f^2 + \frac{v^2}{c^2} = 1$, $f = \sqrt{1-\frac{v^2}{c^2}}$)

Hence $l' = l_0/\gamma$ Lorentz contraction. OK!

Hence Lorentz contraction is consistent with time dilation and the relativity of simultaneity. We see that both observers in relative motion measure the other's stick as contracted, since they observe the ends of the moving rods at different times in the rod's rest frame.

Lorentz transformation

First constructed by Larmor and Lorentz around 1900. Consider S' moving to the right in S with velocity v in the x-direction - Galilei transformation

$$x = x' + vt$$
$$t = t'$$

Consider a relativistic generalization of these. Since x' is measured in S' it is Lorentz contracted to x'/γ:

$$x = x'/\gamma + vt \Rightarrow \boxed{x' = \gamma(x - vt)}$$

Next generalize $t = t'$, to make the velocity of light $= c$ in both frames. This is the central idea in Einstein's derivation. Hence require

$$\left.\begin{array}{l} x = ct, \quad x' = ct' \\ x = x'/\gamma + vt \end{array}\right\} \Rightarrow$$

$$\Rightarrow ct = ct'/\gamma + vx/c \Rightarrow \boxed{t' = \gamma\left(t - vx/c^2\right)}$$

$y = y'$, $z = z'$ are unchanged.

Check the formulas for time dilation and length contraction:

$$\Delta t' = t_2' - t_1' = \gamma\left(t_2 - t_1 - \frac{v}{c^2}\underbrace{(x_2 - x_1)}_{=0 \text{ in the clock's rest frame}}\right) = \gamma\,\underbrace{(t_2 - t_1)}_{=\Delta T}$$

For a rod at rest in S'

$$\underbrace{\Delta x'}_{=l_0} = x_2' - x_1' = \gamma\left(x_2 - x_1 - v\underbrace{(t_2 - t_1)}_{=0 \text{ since the positions are measured at } t_1 = t_2}\right) = \gamma\,(x_2 - x_1) = \gamma\Delta x$$

$\therefore \quad \Delta x = l_0 / \gamma$

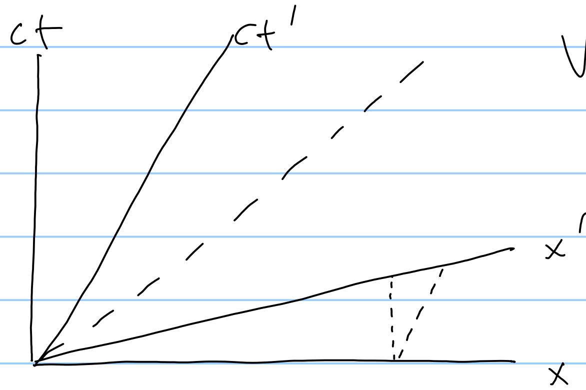
Proper time

Consider a clock at rest in S during a proper time interval $\Delta t = \Delta \tau$.
In S' we get

$$\Delta t' = \frac{\Delta \tau}{\sqrt{1 - \frac{v^2}{c^2}}}$$

while the clock moves a distance

$$\Delta x' = -v \Delta t' = -\frac{v \Delta \tau}{\sqrt{1 - \frac{v^2}{c^2}}}$$



Note that

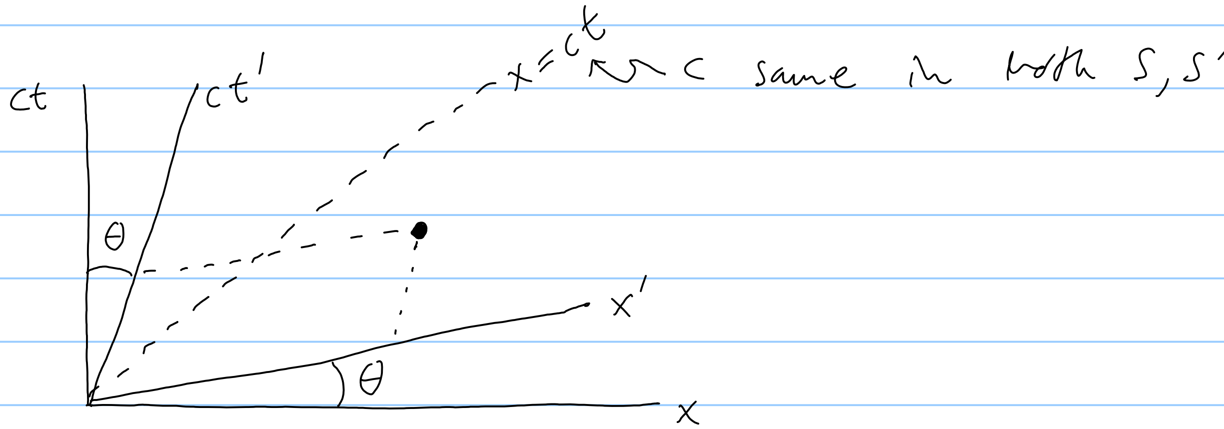
$$c^2 \Delta t'^2 - \Delta x'^2 = \Delta \tau^2 \frac{c^2}{1 - \frac{v^2}{c^2}} - \Delta \tau^2 \frac{v^2}{1 - \frac{v^2}{c^2}}$$

$$\therefore \quad c^2 \Delta \tau^2 = c^2 \Delta t'^2 - \Delta x'^2$$

In general $\Delta s^2 = c^2 \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$

is a <u>Lorentz invariant</u>: the same in all inertial frames

Minkowski diagram

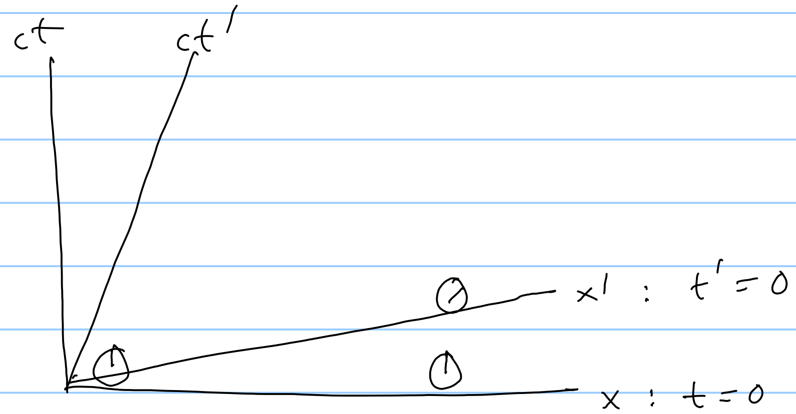


Equation for $x' = 0$: $\quad x = vt \Rightarrow \tan\theta = \frac{x}{ct} = \frac{v}{c}$

Equation for $t' = 0$: $\quad t = vx/c^2 \Rightarrow \tan\theta = \frac{ct}{x} = \frac{v}{c}$

Relative simultaneity

Consider two clocks syncronized in S.
They are <u>not</u> syncronized in S':

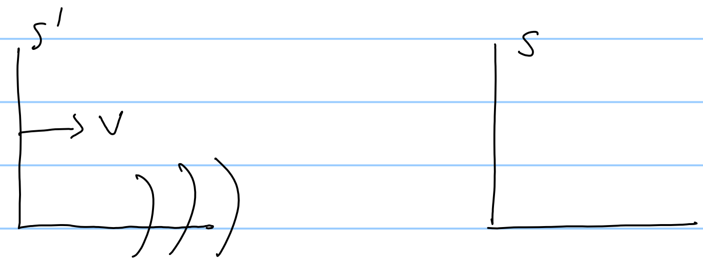


Simultaneous events in S are <u>not</u> simultaneous in S'.

Doppler effect

Consider a light source at rest in S'. S' approaches an observer in S with constant velocity v. Assume that n cycles are observed in time Δt. The length of the approaching wave train is $n \times$ wavelength λ:

$$n\lambda = \underbrace{c\,\Delta t}_{\text{distance light moves}} - \underbrace{v\,\Delta t}_{\text{distance } S' \text{ moves}}$$



Frequency ν : $\nu\lambda = c$ $\Rightarrow$ $n = \nu\left(1 - \frac{v}{c}\right)\Delta t$

Seen from S' the frequency is ν_0 :

$$n = \nu_0 \Delta t' = \nu_0 \Delta t / \gamma$$

$$\therefore \quad \nu_0 \Delta t / \gamma = \nu\left(1 - \frac{v}{c}\right)\Delta t$$

$$\nu = \nu_0 \Big/ \gamma\left(1 - \frac{v}{c}\right) = \nu_0 \frac{\sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{v}{c}}$$

$$= \nu_0 \sqrt{\frac{\left(1 - \frac{v}{c}\right)\left(1 + \frac{v}{c}\right)}{\left(1 - \frac{v}{c}\right)\left(1 - \frac{v}{c}\right)}}$$

$$\boxed{\nu = \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}}\,\nu_0} > \nu_0 \quad \text{(source approaching)}$$

Source leaving is obtained by $v \to -v$:

$$\boxed{\nu = \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}}\,\nu_0} < \nu_0 \quad \text{(source leaving)}$$

Ex Redshift: light from radially leaving stars get the frequency reduced $\Rightarrow$ wavelength longer.

Twin paradox

Consider two twins A and B.
A stays at rest in S while B takes a rocket trip at $\tan\theta = v/c = 4/5$ for 5 years, and then turns back and arrives after 5 more years.

A has aged 10 years but by time dilation, A observes the clock of B runs slow, and B has aged

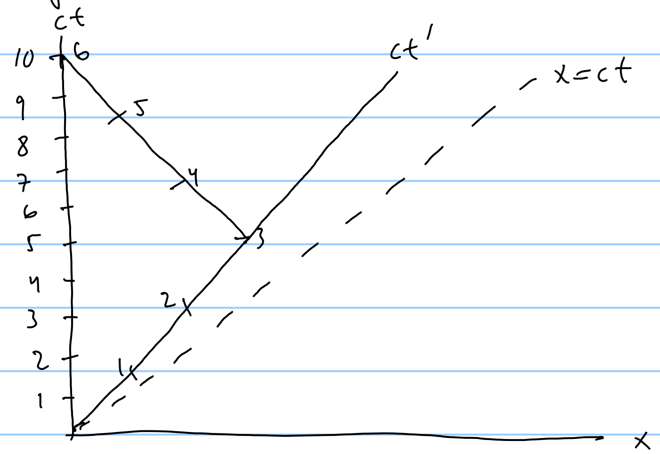
$$\gamma = \frac{1}{\sqrt{1-0.8^2}} = \frac{5}{3} \Rightarrow \Delta t' = \frac{\Delta t}{\gamma} = \frac{10}{5/3} = 6$$

years.

But from the perspective of S', B sees A race away and observes the clock of A to slow down. How can both see the others clock slow down? How can a paradox be avoided when they meet?

The answer is that both agree that B is 4 years younger. We will see why using Minkowski diagrams.

Seen from S :



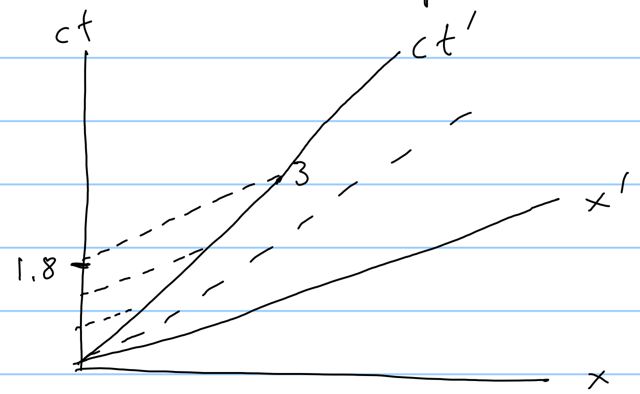
A sees that B ages 6 years.

What about the turnaround point after half the trip? This involves acceleration and the frame B no longer inertial. This difficulty can be avoided by considering two rockets that meet in flight, one at $v = +0.8c$ and one at $v = -0.8c$, and syncronize clocks as they pass.

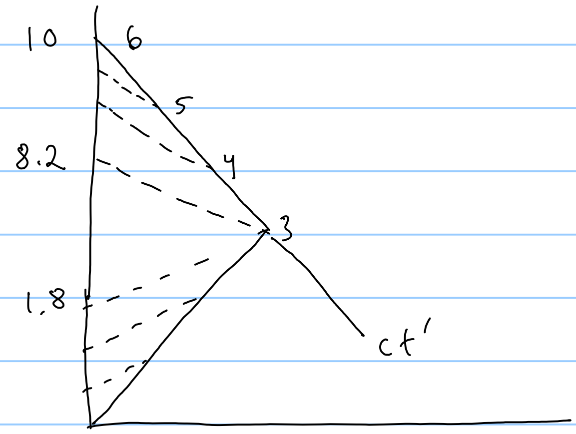
Now consider the trip from the perspective of B. A first leaves at $v = -0.8c$, and B observes A's clock to record that A ages

$$\frac{\Delta t'}{\gamma} = \frac{3}{5/3} = \frac{9}{5} = 1.8 \quad \text{years}$$

on the outbound trip,



At the turnback something special happens. The line of constant t' tilts over since B jumps to a new frame of reference with $v = -0.8c \Rightarrow \tan\theta = -0.8$.
The return trip is then:



The turnaround corresponds to both $ct = 1.8$ l.y. and to 8.2 l.y.
At the turnaround B sees A's clock to jump by $8.2 - 1.8 = 6.4$ years!

Hence both sees the others clock slow down, and both agree that A aged 10 years and B 6 years at the return.

The same conclusion can be reached by studying Doppler shifts.

Velocity addition

Take the Lorentz transformation

$$x' = \gamma(x - vt)$$

$$t' = \gamma(t - vx/c^2)$$

Use $x' = v_p' t'$, $x = v_p t$ $\Rightarrow$

$$v_p' t' = \gamma(v_p t - vt) = \gamma(v_p - v)t$$

$$t' = \gamma(t - v v_p t/c^2) = \gamma\left(1 - \frac{v v_p}{c^2}\right)t$$

Division gives

$$v_p' = \frac{v_p - v}{1 - \frac{v v_p}{c^2}}$$

Solve for v_p:

$$\left(1 - \frac{v v_p}{c^2}\right) v_p' = v_p - v$$

$$v_p' + v = v_p + \frac{v v_p}{c^2} v_p' = v_p\left(1 + \frac{v v_p'}{c^2}\right)$$

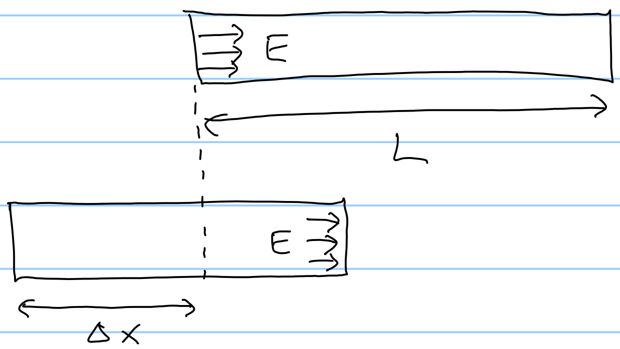
$$\Rightarrow \boxed{v_p = \frac{v_p' + v}{1 + \frac{v v_p'}{c^2}}}$$

Ex: $v_p' = c \Rightarrow v_p = \frac{c + v}{1 + \frac{vc}{c^2}} = c\,\frac{c+v}{c+v} = c$

The speed limit is enforced!

$E = mc^2$ the most well known formula!

Consider a box with mass M and length L. Radiate light with energy E from one end and absorb by the other



We need to know one fact about light: with energy E it carries momentum p: $E = pc$
At emission the box therefore recoils with momentum $p = -E/c$, and the box moves to the left with velocity $v = -E/Mc$ (assuming that M is so large that v is nonrelativistically small). The light reaches the other end in time $\Delta t = L/c$ and is absorbed. Now the box is at rest again, but it has apparently moved a distance

$$\Delta x = v\,\Delta t = -EL/Mc^2$$

But this must be wrong! The center of mass can only move if external forces act. The key observation is that the cm stays still if moving energy E also deposits some mass m on the right end of the box. Condition for not moving the cm:

$$\Delta \bar{x} = 0 = mL + M\Delta x$$

$$\Rightarrow \quad m = -\frac{M}{L}\Delta x = \frac{M}{L}\frac{EL}{Mc^2} = \frac{E}{c^2}$$

$$\Rightarrow \quad \boxed{E = mc^2}$$

The energy carried by the light wave results in a mass increase of $m = E/c^2$ when it is absorbed as heat on the end of the box. Since c^2 is so big, m is very small. But in nuclear reactions and high energy physics the effects can be big.

Relativistic mechanics

The Newton momentum

$$\bar{p} = m\frac{d\bar{x}}{dt} \qquad \text{(Newton)}$$

needs to be generalized relativistically. The numerator $d\bar{x}$ transforms "as it should" relativistically and is left as it is. But dt must be generalized to the invariant $d\tau$. The relativistic momentum then becomes, using $d\tau = \sqrt{1-\frac{v^2}{c^2}}\,dt$

$$\bar{p} = m\frac{d\bar{x}}{d\tau} = \gamma m \bar{v}$$

Newton's law becomes

$$\bar{F} = \dot{\bar{p}}$$

The total energy must be taken as

$$E = mc^2 \frac{dt}{d\tau} = \gamma m c^2$$

to give energy conservation (as is easily shown).

Furthermore:

$$\frac{1}{c^2} E^2 - p^2 = \frac{1}{c^2} \gamma^2 m^2 c^4 - \gamma^2 m^2 v^2$$

$$= \gamma^2 m^2 (c^2 - v^2)$$

$$= \frac{m^2 c^2}{1 - \frac{v^2}{c^2}} \left(1 - \frac{v^2}{c^2}\right) = m^2 c^2$$

$$\therefore \quad E^2 = p^2 c^2 + m^2 c^4$$

If $m = 0$ then we get

$$E = pc$$

which is exactly the energy-momentum relation for light. Massless particles travel at the speed of light!