Generalized Anisotropic Scaling Theory and the Transverse Meissner Transition

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We consider a depinning transition in vortex systems with columnar disorder and tilted applied magnetic fields. From scaling arguments and Monte Carlo simulations, we find that this transverse Meissner transition is governed by a fixed point which is anisotropic in all three directions. This generalization of conventional anisotropic scaling means that the correlation length in different directions diverges with different rates, and we derive exact results for the anisotropy exponents. We make predictions which can be tested in experiments on superconductors with columnar disorder.

Correlated disorder often leads to phase transitions of new universality classes with anisotropic scaling properties. On approaching the phase transition, the correlation length in one certain direction might diverge with a different exponent than in the perpendicular directions [1–3]. Here we present a generalization of anisotropic scaling where all three directions acquire different scaling dimensions, and apply it to the transverse Meissner (TM) transition in superconductors with columnar defects and a magnetic field applied at a finite angle with respect to the columns.

Phase transitions with anisotropic scaling may occur in systems where one direction is singled out, either by applied fields or by correlated disorder. This is, for example, the case for disordered quantum phase transitions since, in a path integral representation of the partition function, static point-correlated disorder maps to perfectly correlated disorder in the imaginary time direction [3]. Another example is given by the Bose glass transitions in superconductors with columnar defects parallel to a magnetic field [4,15,16]. However, analysis of experimental results for the critical behavior has been seriously hampered by the lack of a correct scaling ansatz for the transition. In this Letter we derive scaling properties for the TM transition and obtain a correlation length behaving differently in all three directions, resulting in a correlation volume with shape $\xi^2 \times \xi \times \xi^3$. This scaling behavior is independently confirmed using Monte Carlo simulations. Our results have striking experimental consequences for the scaling of various physical quantities at the transition that are described below.

Scaling analysis.—Our scaling analysis applies to a class of Bose glass models with correlated disorder and finite range interactions between the vortices. We assume that the columnar defects and a strong magnetic field $H_z = H_{\parallel}$ are in the $z$ direction, and a weaker transverse field $H_x = H_{\perp}$ is in the $x$ direction; see Fig. 1.

The presence of a transverse field makes all three directions inequivalent and thus opens the possibility that the correlation length in different directions diverges with different rates as the transition is approached. We therefore make the ansatz $\xi_z = \xi \sim |T - T_c|^\nu$, $\xi_x \sim \xi^x$, $\xi_z \sim \xi^z$. The diverging correlations also lead to critical slowing down with a correlation time $\tau \sim \xi^z$. Accordingly, we make an anisotropic scaling ansatz for the singular part...

FIG. 1 (color online). (a) Vortex lines localized at the columnar defects in the transverse Meissner phase. (b) The vortex liquid phase, where the vortex lines follow the field direction.
of the disorder-averaged free energy density,
\[ f_s(t, h, H_x, H_z) \sim i^{-R+1} f_s(L \{ r \}, L \{ x \}, L \{ y \}, L \{ z \}), \]  
where \( l \) is an arbitrary scale factor, \( t = T - T_c \), and \( h = H_x - H_{c1}^z(T) \). The scaling of the magnetic field as the inverse of length, \( H_1 \sim 1/l_1 \), in Eq. (1) follows from gauge invariance, since it enters as an imaginary vector potential for the bosons [15]. Differentiating Eq. (1) immediately leads to scaling laws for the flux density \( B_i = -\partial f/\partial H_i \) and the magnetic permeability \( \mu_i = \partial B_i/\partial H_i \).

\[ B_x \sim i^{-1+\xi}, \quad \mu_x \sim i^{1-\xi}, \quad \mu_z \sim i^{\xi-1}, \]  

Equating this to zero and identifying the coefficients of \( i \) and \( i^2 \) gives \( B_x = 0 \) and \( B_z = \mu_z H_{c1} \).

This identity together with Eqs. (2) and (3) gives \( \chi = 2 \).

Following Ref. [2] we argue that the magnetic permeability in the \( z \) direction \( \mu_z \) (compressibility in the bosonic language) remains finite at the transition, i.e., \( \mu_z \sim \xi^0 \). This is the case for an ordinary Bose glass transition [2,5] and our Monte Carlo (MC) data support that it is finite also on the tilted transition. Comparing with Eq. (4) then gives the exponent identity \( \xi - \chi = 1 = 0 \). With \( \chi = 2 \) we then obtain \( \chi = 3 \). We call this result 1-2-3 scaling.

\[ \left[ \frac{1}{L^2} \int \frac{d^3 r}{L} \int \frac{d^3 r'}{L} \partial T(r) \Delta T_c(r') \right]^{1/2} \sim \frac{1}{\xi^{(x+1)/2}} \ll |T - T_c| \sim \xi^{-1/\nu}. \]  

Close to \( T_c \) this is true if \( \nu \geq 2/(\chi + 1) = 2/3 \).

**Model and Monte Carlo simulations.**—To calculate \( \nu \) and to test the scaling predictions we perform MC simulations. We use the Bose glass model with an applied transverse magnetic field. The Bose glass partition function is analogous to a path integral for disordered two-dimensional bosons in imaginary time [1,2,5,18]. We use the superconductor language and call the line degrees of freedom a vorticity. This model is relevant for disordered high temperature superconductors far from the upper critical field \( H_{c2} \) [18,19]. The partition function and the Hamiltonian of the short range interaction, lattice version of the model described in Refs. [1,2,18] is

\[ H = J \sum_{r, i=x, y, z} m_i(r)^2 - \sum_r \left[ \mu(x, y) m_z(r) + H_{c1} m_z(r) \right], \]

where we take \( J = 1 \), \( T \) is the temperature, and \( m_i(r) \) is an integer describing the vorticity on the link in direction \( i \) at position \( r \). \( T \) denotes the sum over all possible configurations of integers \( m_i(r) \) on the links of the lattice, which satisfy \( \partial_i m_i = 0 \), i.e., form closed loops. \( \mu(x, y) \) is a random number in the interval \([0, 1]\) that models a strong magnetic field and columnar disorder in the \( z \) direction.

In the MC simulations we mainly study two quantities: the magnetic permeability \( \mu \) and magnetic flux density \( B_i \) in the \( x \) direction, defined as

\[ \langle B_x \rangle = -\frac{\partial f}{\partial H_x} = \frac{1}{L_x L_y L_z} \left[ \left( \sum_r m_i(r) \right) \right], \]

\[ \mu_i = -\frac{\partial^2 f}{\partial H_i^2} = \frac{L_x L_y L_z}{T} \left( \langle B_i^2 \rangle - \langle B_i \rangle^2 \right), \]

where \( \langle \cdots \rangle \) and \( [ \cdots ] \) denote the ensemble average and the disorder average, respectively. To avoid systematic errors \( \langle B_i^2 \rangle \) is calculated using two replicas of the system with the same disorder. Equations (2)–(4) imply the finite size scaling relation for \( \mu_i \) of the form (similar results are obtained for \( B_i \)).

\[ \mu_i(t, L_x, L_y, L_z) \sim L^{2p-1+\xi} \mu_i^{1/\nu} \left[ \frac{L_x}{L} \right]^{1/\nu} \left[ \frac{L_y}{L} \right]^{1/\nu} \left[ \frac{L_z}{L} \right]^{1/\nu}, \]

where \( \mu_i \) is a scaling function and \( p \) depends on the direction \( i \): \( p = 1, \xi \) for \( i = x, y, z \). To extract the critical exponents, we need to use system sizes of shape
$L_x \times L_y \times L_z = aL^x \times L \times bL^z$, where $a$ and $b$ are constants which can be chosen arbitrarily. If $\chi$ and $\zeta$ are chosen correctly, then $\frac{T_L L_z}{L_x} \mu_x$, $\frac{T_L L_z}{L_y} \mu_y$, and $L_y L_z B_x$ become independent of system size at the critical point, since all the arguments in Eq. (11) are then independent of system size. $\chi$ and $\zeta$ are calculated by testing many values and checking for which combination a well defined crossing is obtained for all quantities in all directions. Note that, according to Eq. (4), $\mu_x$ is expected to be a smooth function at $T_c$. This is confirmed in the simulations. Note also that Eq. (6) cannot be expected to hold in detail, since the lattice in the simulated model breaks rotation invariance. However, the exponent $\chi = 2$ is still expected to hold since the critical fixed point is still expected to scale according to Eq. (5).

In order to overcome critical slowing down and to equilibrate the systems close to the transition, we use a “worm” cluster update algorithm [20,21]. We also use the parallel tempering [22] method. We start from an empty system and use 45 000 worm updates for equilibration, followed by equally many updates for measurements. We perform averages over approximately 1000–1500 disorder realizations.

Figure 2 shows the permeability scaling using the 1-2-3 scaling result. It is only when $\chi$ and $\zeta$ are very close to the 1-2-3 result that well defined crossings at the same temperature in both the $x$ and the $y$ directions are obtained. The critical temperature has also been determined from the magnetic flux with the same results. To test universality: Simulations have been performed for both $H_x = 0.1$ and $H_y = 0.2$. A second, weaker, disorder distribution has also been tested. Identical critical exponents, within error bars, are obtained for both values of the transverse field and for both disorder distributions.

We proceed by using Eq. (11) to calculate $\nu$. We note that $\frac{T_L L_z}{L_x} \mu_x$, $\frac{T_L L_z}{L_y} \mu_y$, or $L_y L_z B_x$, for different system sizes can be collapsed onto single curves close to $T_c$, if the correct value of $\nu$ and $T_c$ are used. The values of $\nu$ and $T_c$ are best found by a multiparameter minimization of

$$\sum_L \int (\mathcal{O}_L(x) - \mathcal{O}(x))^2 dx,$$

where $\mathcal{O}_L = \frac{T_L L_z}{L_x} \mu_x$, $\frac{T_L L_z}{L_y} \mu_y$, or $L_y L_z B_x$, and $x = (T - T_c) a L^{1/\nu}$. Here $\mathcal{O}(x)$ is obtained by linear interpolation of the data points for a given system size, and $\mathcal{O}(x)$is its mean value. We have tested various other interpolation methods, all giving the same values for $T_c$ and $\nu$, within error bars. Figure 3 displays the collapse resulting from minimization of Eq. (12) with $\mathcal{O} = \frac{T_L L_z}{L_x} \mu_x$ and $\frac{T_L L_z}{L_y} \mu_y$ for $H_\perp = 0.1$. The best fit is obtained for $\nu = 0.7 \pm 0.1$ for both the magnetic fields that we tried ($H_\perp = 0.1, 0.2$) in both the $x$ and the $y$ directions. The error bars are estimated using the bootstrap method [23], by varying the range of system sizes included in the fit and by varying the temperature interval in which the fit is made. In Fig. 4 the same method is used to collapse the total flux curves. The result for $T_c$ is within error bars the same as $T_c$ estimated above. The value of $\nu$ is estimated to $\nu = 0.68 \pm 0.06$. This value of $\nu$ is very close to the Harris criterion bound, given above ($\nu \geq 2/3$).

**Implications for experiments.**—The anisotropic 1-2-3 scaling predicts a number of experimental testable results for the transverse Meissner transition. The magnetic response of the system at the transition is summarized in the scaling relations for the magnetic permeability in Eqs. (2) and (3).

We also derive scaling relations for the resistivity and the $I-V$ characteristics. We assume that the transition is
and $E_i$ anisotropic in one direction only. Our theory makes distinct
predictions that would be interesting to test experimentally on superconductors with columnar defects.

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Consider the linear resistivity $\rho_{ij} = \frac{\partial E_i}{\partial J_j}$. According to
the scaling relations above and 1-2-3 scaling, the $z$ independent resistivity ratios scale as $\rho_{xx} \sim \xi^2 \sim (T - T_c)^{-2\nu}$, $\rho_{xz} \sim \xi^4 \sim (T - T_c)^{-4\nu}$, and $\rho_{zz} \sim \xi^{-2} \sim (T - T_c)^{2\nu}$. The $I$-$V$ characteristics, right at the critical temperature, is predicted to be $E_x \sim J^{(z+1)/\xi^2}$, $E_y \sim J^{(z+1)/\xi^2}$, and $E_z \sim J^{(z+1)/(1+\xi)}$ in the respective direction. Note that all
these relations, which apply at finite tilt, are different from the corresponding relations for the Bose glass transition at zero tilt.

In summary, we determine the critical properties of the TM transition from exact scaling arguments, and from
numerical Monte Carlo simulations. The divergence of the correlation volume at the TM transition is anisotropic
in all three directions, given by $\xi^2 \times \xi \times \xi^3$, with $\xi \sim |T - T_c|^{-\nu}$, $\nu = 0.68 \pm 0.06$. This is very different from the Bose glass transition, where the correlation length is anisotropic in one direction only. Our theory makes distinct
predictions that would be interesting to test experimentally on superconductors with columnar defects.

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