Universal Conductivity of Dirty Bosons at the Superconductor-Insulator Transition

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(Received 21 April 1992)

The superconductor-insulator transition at zero temperature in two dimensions is studied by Monte Carlo simulations of interacting bosons (Cooper pairs) moving in a quenched random potential. We calculate the universal conductivity σ^* and the critical exponents at the superconductor-insulator transition. For the short-range repulsive case we find $\sigma^* = (0.14 \pm 0.03)\sigma_Q$, where $\sigma_Q^{-1} \equiv R_Q \equiv h/(2e)^2 \simeq 6.45$ kΩ, and for long-range Coulomb interactions we find $\sigma^* = (0.55 \pm 0.1)\sigma_Q$.

PACS numbers: 74.65.+n, 74.70.Mq, 74.75.+t

The localization transition at zero temperature has attracted much attention over the years. For fermionic systems this is the metal-insulator transition. In bosonic systems, the conducting phase is generally superconducting at zero temperature. Recent experiments on various 2D systems have convincingly demonstrated the possibility of a continuous $T = 0$ quantum phase transition between a superconducting and an insulating phase. Examples of such systems are metallic films [1–3], high-T_c films [4], and Josephson-junction arrays [5].

One can tune through the superconductor-insulator transition by systematically varying a control parameter δ, which can, for example, be the film thickness (which effectively controls the strength of disorder in the system) or the strength of an applied magnetic field. In the case of truly amorphous films, the film is insulating for small δ. As δ is increased the film becomes superconducting at a critical but nonuniversal value δ_c. Experiments [1, 2] suggest that the conductivity σ^* right at the transition is finite and possibly *universal* with a value close to the quantum of conductance $\sigma_Q \equiv (2e)^2/h$, where $2e$ is the Cooper pair charge, although in most cases it has not been definitively established that the experiments are able to approach the critical regime.

In this paper we study the universal properties of the superconductor-insulator transition in systems modeled as interacting dirty bosons [6–13]. We calculate the universal conductivity σ^* by Monte Carlo (MC) simulations for two distinct universality classes. For the case of short-range repulsive interactions, we find the result $\sigma^* = (0.14 \pm 0.03)\sigma_Q$. A similar value has been found by Runge using the technique of exact diagonalization for small lattices [11]. We have also performed the first MC calculation for the case of long-range Coulomb interactions. Here we find $\sigma^* = (0.55 \pm 0.1)\sigma_Q$.

We adopt a boson picture to describe the superconductor-insulator transition [7]. In a superconductor we cannot neglect the attractive pairing interaction among the fermions which produces an excitation gap. We assume that there are no low-energy fermionic degrees of freedom present throughout the transition, but only bosonic degrees of freedom (Cooper pairs). This hypothesis applies to Josephson-junction arrays and granular films, but is more controversial for homogeneous films. However, Hebard and Paalanen [3] have convincingly argued from transport data on InO films that the fermion gap remains finite at the critical point in this system.

It is well known [7] that at integer filling and neglecting Coulomb interactions and disorder the 2D boson Hubbard model can be approximately mapped onto the (2+1)D classical XY model, without changing the universality class. The XY model is given by the Hamiltonian $H = -K \sum_{<l,l'>} \cos(\theta_l - \theta_{l'})$, where l and l' are nearest-neighbor sites, and θ is the phase of the order parameter in the path integral representation. Unfortunately, away from integer filling, the same mapping gives *complex* weights in the equivalent classical problem [14]. However, the Villain [15] form of the XY model, in which the cosine is replaced by a periodic Gaussian, can be rewritten in terms of integer currents $\mathbf{J} = (J^x, J^y, J^\tau)$ on the links of the (2+1)D lattice, with the Hamiltonian [16]

$$H = \frac{1}{2K} \sum_{\mathbf{r},\tau} \mathbf{J}^2(\mathbf{r}, \tau), \tag{1}$$

where $\mathbf{r} = (x, y)$ and τ is the imaginary time. Note that J^τ is just the particle density. The integer-link variables (J^x, J^y, J^τ) take integer values from $-\infty$ to ∞, subject to the divergence-free constraint given by the continuity equation, $\nabla \cdot \mathbf{J} = 0$.

One can now generalize Eq. (1) to noninteger fillings, and include disorder and Coulomb interactions. We thus arrive at a purely *real* Hamiltonian

$$H = \frac{1}{K} \sum_{\mathbf{r},\tau} \{\tfrac{1}{2}\mathbf{J}^2(\mathbf{r}, \tau) - [\mu + v(\mathbf{r})] J^\tau(\mathbf{r}, \tau)\} + \frac{e^{*2}}{2K} \sum_{\mathbf{r},\mathbf{r}',\tau} [J^\tau(\mathbf{r}, \tau) - n_0] G(\mathbf{r} - \mathbf{r}') [J^\tau(\mathbf{r}', \tau) - n_0], \tag{2}$$

where G is the $1/r$ Coulomb potential modified for the periodic lattice, μ is the chemical potential of the bosons, and $v(\mathbf{r})$ is a random site energy with uniform distribution between $-\Delta$ and Δ. We include Coulomb interactions because they change the universality class [6] and because they correspond to the experimental situation.

In this paper we study Eq. (2) by MC simulations for the parameter values $e^{*2} = 0$ and $e^{*2} = 0.5$, and $\Delta = 0.5$. We also took $\mu = 0.5$ (so the average boson density is $n_0 = \frac{1}{2}$) in order that the system is far away in parameter space from a Mott insulating phase [7] which could give unwanted crossover effects. We use simple cubic lattices of size $L \times L \times L_\tau$ with periodic boundary conditions in all directions. L_τ is related to the temperature of the bosons by $L_\tau = \hbar/k_B T$. We use two different MC moves that fulfill the divergence-free condition $\nabla \cdot \mathbf{J} = 0$: (1) closed local current loops in the three planes, and (2) global current lines that go straight across the system in three directions $\delta = x, y, \tau$. We perform a "thermal" MC average for a given realization of the disorder, denoted by $< \cdots >$, as well as an average over disorder realizations, indicated by $[\cdots]_{\rm av}$.

Our strategy for the simulation of the quenched disorder is to average over many disorder realizations on small systems. This method has proved to be the key to successful analysis of the spin-glass problem [17]. We have to do many disorder realizations since the fluctuations in the conductivity between different disorder realizations is of the same order of magnitude as the mean conductivity itself [18] (as in universal conductance fluctuations in free fermion systems). At the critical point, for each disorder realization, we typically do 2000 to 10 000 equilibration sweeps (one sweep is defined here as one sweep through the lattice with local updates plus one sweep with global updates) followed by equally many measurements with from 1 to 10 sweeps skipped between every measurement. We do between 500 and 1000 disorder realizations at the critical point, and fewer away from the critical point. Two replicas are run in parallel for each disorder realization. In this way the equilibration can be checked by standard techniques [17] by monitoring the overlap between the two replicas.

We locate the critical point by finite-size scaling of the frequency-dependent stiffness, $\rho(i\omega_n)$, defined as

$$\rho(i\omega_n) = \frac{1}{L^2 L_\tau}\left[\left\langle\left|\sum_{\mathbf{r},\tau} e^{i\omega_n \tau} J^x(\mathbf{r},\tau)\right|^2\right\rangle\right]_{\rm av}, \qquad (3)$$

where $\omega_n = 2\pi n/L_\tau$ is the nth Matsubara frequency. The zero frequency component will obey the following finite-size scaling ansatz [7]:

$$\rho(0) = L^{2-d-z}\tilde{\rho}(bL^{1/\nu}\delta, L_\tau/L^z), \qquad (4)$$

where L is the spatial system size, d is the dimension of the quantum problem ($d = 2$ here), $\tilde{\rho}$ is a universal function, b is a nonuniversal scale factor, $\delta = (K - K^*)/K^*$, and z is the dynamical exponent defined as the ratio of the critical exponents for the correlation length in the time and space direction. At the critical point, $L^z\rho(0)$ is a function of L_τ/L^z, so plots of $L^z\rho(0)$ for different system sizes will intersect at the transition only if L_τ/L^z, the effective aspect ratio of the sample, is kept constant. It is thus necessary to make an initial guess for z [7, 9].

It has been argued that $z = 2$ for the short-range case [8], and $z = 1$ [6] in the long-range case [19], so we take these values as our initial choice. In Fig. 1 our results are shown for $L^2\rho(0)$, with $e^{*2} = 0$. The data for different L cross at a single point, which determines the critical coupling to $K^* = 0.248 \pm 0.002$. The error bars are quite small at the transition because we invested more computer time there. The compressibility remained finite through the transition, which is consistent with a transition to an insulating Bose-glass phase. With $e^{*2} = 0.5$ we find $K^* = 0.240 \pm 0.003$, from the intersection of the data for $L\rho(0)$, as shown in Fig. 2. For significantly different choices of z, the data did not scale in either case.

We have calculated the critical exponents from the single-particle correlation functions in the time and space directions [20]. For the case of short-range interactions, $e^{*2} = 0$, we find the following exponents: $\nu = 1.0 \pm 0.1$, $\eta = -0.10 \pm 0.15$, and $z = 2.0 \pm 0.1$; the errors come mainly from the uncertainty in the value of K^*. These exponents are consistent with the values $\nu \geq 1$, $\eta \leq 0$, and $z = 2$ suggested by Fisher *et al.* [8], and with the results obtained by Runge [11]. With long-range interactions, $e^{*2} = 0.5$, we obtain $\nu \simeq 1.0$, $\eta \simeq 0.9$, and $z \simeq 1.0$. The value of z agrees with Ref. [8] while ν and η have not been calculated before. Note that the values of z for both cases agree with the initial guess used to determine the shape of the lattices.

The conductivity is given by the Kubo formula [7]

$$\sigma(i\omega_n) = 2\pi\sigma_Q \frac{\rho(i\omega_n)}{\omega_n}, \qquad (5)$$

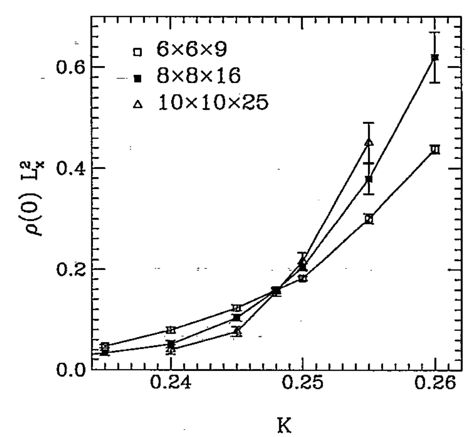

FIG. 1. Determination of the critical coupling K^* for short-range interactions, $e^{*2} = 0$. Plotted are Monte Carlo data for $L^2\rho(0)$, where $\rho(0)$ is the stiffness, vs coupling constant K. The lattice sizes are $L \times L \times L^2/4$. Error bars on the MC points correspond to a statistical error of 1 standard deviation. According to finite-size scaling (see text) the critical point is located where the curves cross. We find $K^* = 0.248 \pm 0.002$.

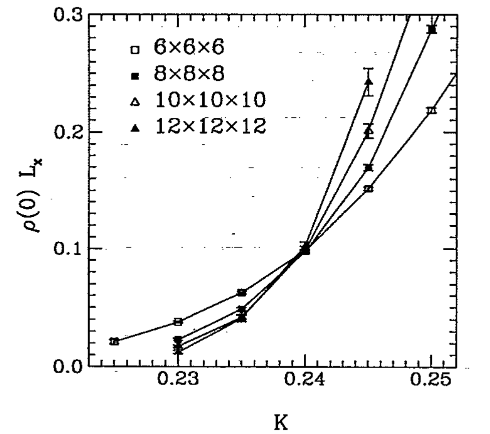

FIG. 2. Determination of the critical coupling K^* for long-range interactions, $e^{*2} = 0.5$. Plotted are Monte Carlo data for $L\rho(0)$, where $\rho(0)$ is the stiffness, vs coupling constant K. The lattice sizes are $L \times L \times L$. Error bars on the MC points correspond to a statistical error of 1 standard deviation. According to finite-size scaling (see text) the critical point is located where the curves cross. We find $K^* = 0.240 \pm 0.003$.

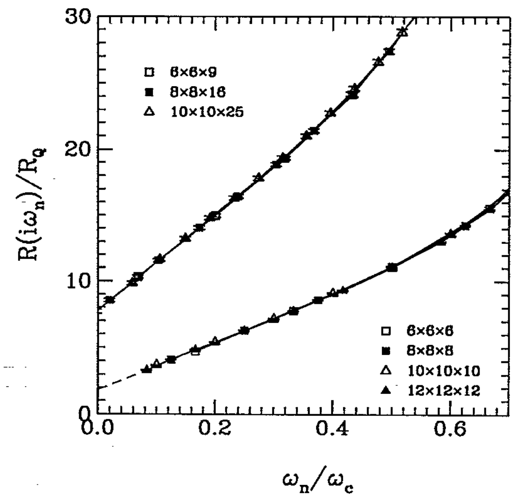

FIG. 3. Determination of the universal conductivity σ^* at the superconductor-insulator transition. Plotted are MC data for the resistance $R(i\omega_n) \equiv 1/\sigma(i\omega_n)$ at the nth Matsubara frequency ω_n vs ω_n/ω_c, where ω_c is the ultraviolet cutoff frequency set by the lattice spacing [22]. The upper set of data are for short-range interactions, and the lower set of data are for long-range interactions with $e^{*2} = 0.5$. The lattice sizes are $L \times L \times L^2/4$ in the short-range case and $L \times L \times L$ in the long-range case. Error bars on the MC points correspond to 1 standard deviation. The data collapse on lattice-size-independent curves. Linear extrapolation to $\omega_n = 0$ (dashed line) is shown for the long-range case.

where $\rho(i\omega_n)$ is the frequency-dependent stiffness given in Eq. (3). Close to the transition $\rho(\omega) = \xi^{2-d}\xi_\tau^{-1} \times\widetilde{\rho}(\omega/\Omega)$ [6], where $\Omega \sim \xi_\tau^{-1}$ is a characteristic frequency. Requiring that $\rho(\omega)$ remain finite at the transition, $\widetilde{\rho}$ must scale as $\widetilde{\rho}(x) = Cx^{(d-2+z)/z}$ as $x \to 0$, where, by two-scale factor universality [21], C is a *universal* constant. Thus, from Eq. (5), we see that $\sigma(\omega \to 0)$, in $d = 2$, is *universal* at the transition.

In Fig. 3 we show MC data for the conductivity in the short-range case, $e^{*2} = 0$, and in the long-range case, $e^{*2} = 0.5$. Plotted is the resistance $R \equiv 1/\sigma$ (in units of R_Q) vs ω_n/ω_c, where ω_c is the ultraviolet frequency cutoff equal to the inverse of the lattice spacing in the imaginary time direction [22]. In both cases the data collapse on curves that are independent of the lattice size. At low frequencies ω_n the curves become linear, i.e., $\sigma(i\omega_n) = \sigma^*/(1 + |\omega_n|\tau_c)$, where $\tau_c \sim 1/\omega_c$ is a nonuniversal relaxation time controlled in our model by the ultraviolet cutoff. This is easily seen to analytically continue to the Drude form of the conductivity $\sigma(\omega + i\delta) = \sigma^*/(1 - i\omega\tau)$, which extrapolates to σ^* at zero frequency. Based on all our results, incorporating two aspect ratios for the short-range case and two values of e^{*2} for the long-range case, we find for the short-range case $\sigma^* = (0.14 \pm 0.03)\sigma_Q$, and in the long-range case $\sigma^* = (0.55 \pm 0.1)\sigma_Q$. The error estimates are dominated by the statistical uncertainty in the critical coupling K^*.

We now briefly discuss our results. In the short-range case, we find the rather small value $\sigma^* \simeq 0.14\sigma_Q$. The universal conductivity has previously been calculated for the case where the disorder is neglected [7]. In this case, the superconductor-insulator transition is a Mott-Hubbard transition with $\sigma^* \simeq 0.285\sigma_Q$. Thus we find that even though the present model is somewhat more realistic, including the disorder takes us *further* from the experimental value. In 1D a suitably defined universal conductance can be calculated exactly with and without disorder [7]. This solution shows that the ratio of σ^* in the dirty case and in the pure case is 3/4, which is of the same order of magnitude as the ratio $0.14/0.285 \simeq 0.5$ between the MC results in 2D.

In the long-range case we find $\sigma^* \simeq 0.55\sigma_Q$, which is significantly increased with respect to the short-range case. The probable reason for this increase is that with long-range $1/r$ interactions the system is closer to being self-dual at the critical point [7] and a self-dual system would have $\sigma^* = \sigma_Q$. Experiments on ultrathin superconducting films suggest that σ^* is of the same order of magnitude as σ_Q, although low enough temperatures have probably not yet been reached for a decisive comparison with our value $\sigma^* \simeq 0.55\sigma_Q$.

As a test of the universality we have tried reducing the density of bosons by a factor of 2, changing the ratio of the time and space dimensions, and reducing the strength of the Coulomb interaction by a factor of 2. We also tried two different modifications of the Coulomb potential for the periodic lattice. None of these changes affected σ^* [20].

We conclude with some additional remarks about experiment. Real experiments are carried out in the limit where the system size, L, is much bigger than the phase coherence length, $\xi \sim T^{-1/z}$. In this limit one should view the measured conductivity as arising from *incoherent* self-averaging of domains in the sample whose dimensions are the phase coherence length. In the opposite limit, $L \ll T^{-1/z}$, we expect large variations in the conductance from sample to sample, and the average over samples will not necessarily be the same as the value obtained for $L \gg T^{-1/z}$. In general, the average conductivity at zero frequency but finite temperature T is given by a scaling function $\widetilde{\sigma}(1/TL^z)$. The experimental results are for $\widetilde{\sigma}(0)$. Since $T = 1/L_\tau$, the argument of the scaling function can be reexpressed as L_τ/L^z, which is the aspect ratio of the sample and is therefore *finite* in our simulations. It is more difficult to obtain the full scaling function and the experimental quantity $\widetilde{\sigma}(0)$, but an attempt to determine these quantities is underway.

The guiding light behind this work is M. P. A. Fisher whom we thank for a large number of extremely valuable suggestions and discussions. We also acknowledge discussions with Peter Weichman and Andrew Millis. We thank Professor J. C. Swihart for computational assistance and Professor H.-O. Meyer for use of the SEMELE computer facility. A.P.Y. and E.S.S. acknowledge the kind hospitality of Ecole Normale Supérieure, Paris, and CEA, Saclay, where a major part of this work was done, as well as generous allocations of computer time at CCVR, Saclay, and the San Diego Supercomputer Center. M.W. is supported by NFR Grant F-DP 9901-302. S.M.G. is supported by DOE Grant DE-FG02-90ER45427 and NCSA DMR-910014N. A.P.Y. and E.S.S. are supported by NSF DMR-91-11576.

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[22] Actually for the case of short-range interactions the data are plotted against the scaling variable $x \equiv \omega_n/\omega_c - \alpha L^{-2}(\omega_c/\omega_n)$, where $\alpha = 0.17$ is a correction factor due to the finite sample size. The data scale fairly well even with $\alpha = 0$ and give nearly the same estimate for σ^*. See Refs. [7, 20] for a discussion. For the long-range case, the best fit was with $\alpha = 0$.