Ratchet Effect for Cold Atoms in an Optical Lattice

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The possibility of realizing a directed current for a quantum particle in a flashing asymmetric potential is investigated. It is found that quantum resonances, where the value of the effective Planck constant is equal to an integer or half-integer multiple of π, give rise to a directed current. The effect should be readily observable in experiments.

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The study of nanoscale quantum pumps is of considerable current interest, motivated by the quest to explore basic quantum phenomena, by recent significant experimental progress, and also by the potential practical implications in terms of artificial devices [1]. Ratchet models have been introduced and studied for processes that allow extraction of work from diffusive motion without application of macroscopic gradients. For example, ratchets have been discussed in the context of biological motors [2] and for manipulating magnetic flux in artificially nanostructured superconductors [3].

The extension of ratchet models to the quantum regime has attracted considerable attention lately. Rocking ratchets, where an alternating bias field is imposed on an asymmetric periodic potential, were studied in the presence of noise in Refs. [4,5]. These predict a current reversal due to quantum effects which was observed experimentally for electrons in heterostructures in Ref. [6]. With the advent of optical lattices, the possibility has opened up to study dynamics of systems of quantum particles in periodic potentials in the *absence* of noise. These provide a novel setting for ratchet effects, and interest in coherent quantum ratchets has therefore grown recently. This research is closely connected to the study of the quantum kicked rotor, which can be experimentally realized in a *symmetric* optical-lattice potential that is flashed on and off periodically in time. Enhanced spreading in momentum space due to quantum resonances at certain parameter values [7,8] were observed experimentally in Refs. [9,10]. A quantum flashing ratchet, which we study here, can be seen as a generalization of the kicked rotor to an asymmetric potential. Reference [11] found theoretically that no long-time transport was present for strictly periodic flashing. A scheme with unequal spacing between the flashes was, however, shown to result in transport. Similar schemes were studied experimentally in Refs. [12,13]. Further variations include chaos-assisted quantum transport for Hamiltonian systems with an asymmetric, mixed phase space [14]. Transport was also observed experimentally in an optical-lattice setup that utilizes oscillations between internal states of the atoms to create a ratchet effect [15].

In this Letter we study a nonlinear quantum pump acting on a particle in an optical lattice, driven by a flashing ratchet mechanism. The focus of this Letter is to understand how quantum resonances can be utilized to accomplish directed transport. By numerical and analytical means we study the influence of parameter values and initial conditions on the transport properties of these systems and map out the main resonances.

System.—Consider a particle in one dimension subject to the Schrödinger equation [8–11]

$$i\bar{k}\frac{\partial \psi}{\partial t} = -\frac{\bar{k}^2}{2}\frac{\partial^2 \psi}{\partial x^2} + Kv(x)\sum_{n=0}^{\infty}\delta(t-n)\psi, \tag{1}$$

where $v(x)$ is assumed to be a sawtooth-shaped ratchet potential. In the case of a realistic atomic system, two standing laser waves can be combined to form the potential

$$v(x) = \sin x + \alpha \sin 2x, \tag{2}$$

where the parameter α controls the skewness of the periodic potential that is the origin of the ratchet effect. A value of α between 0 and 0.5 approximates a sawtooth potential where the sawteeth lean to the left; this is expected to yield a positive current in the usual case of classical diffusive motion. The profile is more asymmetric for larger values of α. The potential is assumed to be flashed on and off at periodic intervals. To facilitate calculations we have taken the pulses to be delta functions in time. The case $\alpha = 0$, corresponding to a pure sine potential, is formally equivalent to the kicked rotor, as mentioned above. The addition of a second sine wave adds complexity to this classic problem and opens the possibility of a ratchet effect [11].

The units in Eq. (1) are chosen such that the temporal period of the flashing and the spatial period of the lattice are unity. In terms of physical quantities, the effective Planck constant is

$$\bar{k} = 8\omega_R T, \tag{3}$$

where T is the period of the flashing and $\omega_R = \hbar k_L^2/2m$ is the recoil frequency of the applied laser field, with k_L the photon wave number giving a lattice period $(2k_L)^{-1}$ for the optical potential felt by the atoms. The effective potential strength is

$$K = \bar{k}\frac{TE_{\rm OL}}{\hbar} = \bar{k}P, \tag{4}$$

where $E_{\rm OL}$ is the amplitude of the periodic potential. We have implicitly defined the auxiliary quantity $P = K/\bar{k}$ in Eq. (4), because it makes the notation simpler and, in addition, P has a simple interpretation: it is just the ratio between the potential strength and the kicking frequency.

When the potential flashes are assumed to be delta-function kicks, the time evolution can in principle be described analytically: in operator language, the time development for one period of the flashing is

$$|\psi(t)\rangle = e^{-i\bar{k}\hat{k}^2/2}e^{-iPv(\hat{x})}|\psi(t-1)\rangle, \tag{5}$$

where $\hat{k} = -i\partial/\partial x$. In momentum space the wave function at time t is given by

$$\psi(k,t) = e^{-i\bar{k}k^2/2}\sum_{q=-\infty}^{\infty}F(k-q)\psi(q,t-1), \tag{6}$$

where the function F is the Fourier transform of the phase factor $e^{-iPv(x)}$. Although the analytical formula, Eq. (6), is useful for studying analytical properties, it is easier to solve for the time evolution numerically using a fast Fourier transform (FFT) algorithm. We assume the wave function to be periodic with the period of the potential. The wave function is propagated in time, one period in each step, by first multiplying the wave function components in k space by a phase factor $\exp(-i\bar{k}k^2/2)$, then Fourier transforming the wave function and applying the phase factor associated with the potential, $\exp(-iPv(x))$, to the real-space wave function before Fourier transforming back. In most of this Letter we assume an initially homogeneous wave function with zero momentum; this is also a good approximation for a wave packet that extends over many lattice sites. We also study the case where the initial state is an eigenstate of the static potential.

Time evolution.—Figure 1 shows a few examples of the time dependence of the expectation of the wave number, $\langle k\rangle = \langle\psi(t)|k|\psi(t)\rangle$.

We have chosen the potential asymmetry to be $\alpha = 0.3$ in all examples. The main features of the wave number vs Planck constant curve do not depend critically on the value of α, but as a rule the drift is stronger for larger α. In all examples, the initial state has been taken to be the uniform zero-momentum state. We discuss other initial states below. The topmost two panels display a generic nontransporting case. Two examples for the effective Planck constant are shown, $\bar{k} = 0.711\pi$ and $\bar{k} = 0.232\pi$. There is no ratchet effect here: the momentum averages out to zero at long times, giving zero drift. This is consistent with the findings of Ref. [11], where it was shown that for generic choices of parameters $\bar{k}$ and P, there is no long-time transport in the absence of noise. This is expected since the classical phase space is symmetric, and transport is absent also in the classical case. For the present case of a rather weak potential, $P = 0.5$, the mean squared momentum immediately reaches its peak value and stays bounded, as can be seen in the upper right panel of Fig. 1. The

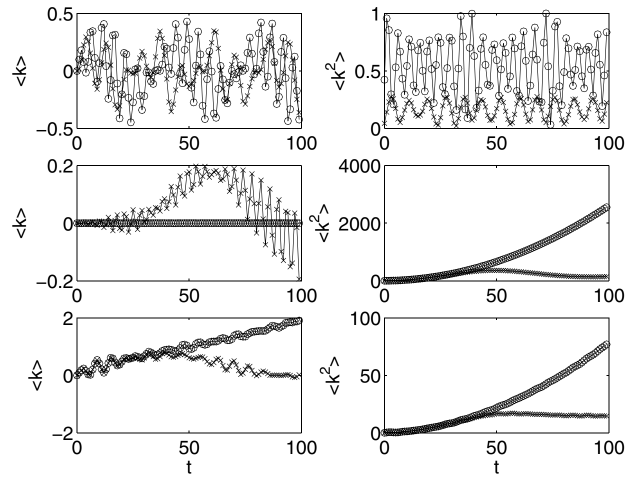

FIG. 1. Time development of the wave number $\langle k\rangle$ and the square wave number $\langle k^2\rangle$ for a number of parameter choices. In all cases, the potential strength $P = 0.5$ and the asymmetry $\alpha = 0.3$. In the topmost two panels, the lines with circles represent the choice $\bar{k} = 0.711\pi$ and the lines with crosses are $\bar{k} = 0.232\pi$. In the middle panels, the lines with circles represent $\bar{k} = 1.0\pi$ and the lines with crosses $\bar{k} = 1.001\pi$. In the bottom two panels, the lines with circles represent $\bar{k} = 0.5\pi$ and the lines with crosses $\bar{k} = 0.501\pi$.

complementary case of a strong potential was studied in Ref. [11].

Resonances.—The picture is totally different in the case of quantum resonances. Because the periodicity of the potential restricts the wave number k to integer values, the kinetic-energy phase factor $e^{-i\bar{k}k^2/2}$ in Eqs. (5) and (6) behaves differently when $\bar{k}$ is a rational multiple of π. This is a long studied subject in the case of the quantum kicked rotor [7–10], where quantum resonances are seen to lead to a quadratic increase of the kinetic energy with time. In the case of a ratchet potential, these resonances open up for the possibility of directed transport, i.e., a genuine ratchet effect, as we now see. The two middle panels of Fig. 1 show the behavior of the system close to the resonance at $\bar{k} = \pi$. Not only is a ratchet effect absent, but the mean momentum of the wave packet is zero at all times. However, the mean *squared* momentum increases quadratically with time. The wave packet thus spreads equally fast towards positive and negative momenta because of the resonance. Included is also the result for the slightly off-resonance case $\bar{k} = 1.001\pi$. The momentum fluctuates about zero, although the amplitude is small because of the vicinity to the resonance. The departure from resonant behavior is even more evident in the curve for the squared momentum, which starts to fluctuate about a positive mean value after a few tens of periods. In the bottom panels of Fig. 1, we display the time evolution of the momentum at $\bar{k} = \pi/2$. At this resonance, for a zero-momentum initial state, there is indeed a linear increase in momentum with time. This is finally a true ratchet effect, although it is driven not by noise, but by a quantum resonance. For

$\bar{k} = 0.501\pi$, the departure from resonant behavior after some time is clearly visible.

Let us now quantify the resonant behavior. In the simplest case, $\bar{k} = n \times 4\pi$, we have that the kinetic part of the time evolution operator is $e^{-i\bar{k}k^2/2} = 1$ for all k. Between two kicks, the system returns exactly to the state it was in just after the last kick, and the wave function at integer times t, i.e., just after each kick, is given by

$$\psi(x, t) = e^{-itPv(x)}\psi(x, 0). \tag{7}$$

This leads to the wave number expectation

$$\langle k \rangle = -tP \int dx v'(x)|\psi(x, 0)|^2, \tag{8}$$

which is zero if the initial state is a plane wave, but generally, if the integral in Eq. (8) is nonzero, then the momentum will increase linearly.

The resonances at $\bar{k} = n \times 2\pi$, with n odd, are also quite easily dealt with. Note that now $e^{-i\bar{k}k^2/2} = (-1)^{k^2}$, but $(-1)^{k^2} = (-1)^k$, and furthermore

$$\sum_{k=-\infty}^{\infty} e^{-ikx}(-1)^k = \delta(x - \pi), \tag{9}$$

where the delta function is periodic with period 2π. This leads to a time development

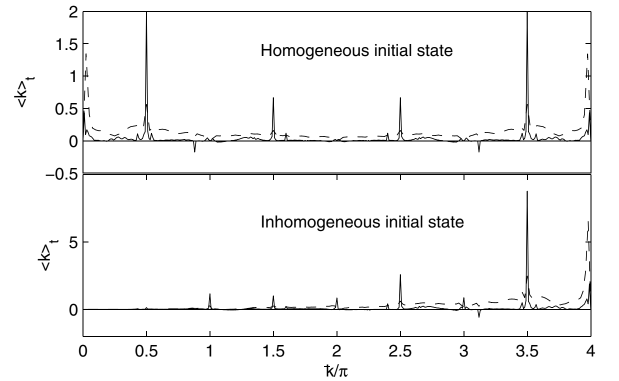

FIG. 2. Mean wave number of the wave packet over time, as a function of the effective Planck constant $\bar{k}$. The asymmetry of the potential has been chosen to $\alpha = 0.3$, and the potential strength $P = 0.5$. In the topmost panel, the initial state is a homogeneous zero-momentum state and the averaging is done over 20 temporal periods for the dashed line and 100 periods for the solid line. In the lower panel, the initial state was chosen to be the ground state of the potential.

$$\psi(x, t) = e^{-iPv(x+\pi)}\psi(x + \pi, t - 1), \tag{10}$$

and thus the general time evolution is

$$\psi(x, t) = \begin{cases} e^{-i(t/2)P[v(x+\pi)+v(x)]}\psi(x, 0), & t \text{ even}, \\ e^{-iPv(x+\pi)-i[(t-1)/2]P[v(x+\pi)+v(x)]}\psi(x + \pi, 0), & t \text{ odd}. \end{cases} \tag{11}$$

This leads, again, to the conclusion that the mean momentum stays zero if the initial state is homogeneous, but increases linearly otherwise.

For successive resonances, the analysis becomes more complicated, but for $\bar{k}$ equal to an odd multiple of π, the time evolution can be solved using Floquet analysis [16], and it can again be shown that the drift is zero for an initially homogeneous state. The time evolution for the case $\bar{k} = \pi$ is

$$\psi(x, t+1) = \frac{1-i}{2}e^{-iPv(x)}\psi(x, t) + \frac{1+i}{2}e^{-iPv(x+\pi)} \times \psi(x + \pi, t), \tag{12}$$

while that at $\bar{k} = \pi/2$ goes as

$$\psi(x, t+1) = \frac{1}{2}\bigg[\sqrt{-i}e^{-iPv(x)}\psi(x, t) + e^{-iPv(x)}\psi\left(x + \frac{\pi}{2}, t\right) - \sqrt{-i}e^{-iPv(x+\pi)}\psi(x + \pi, t) + e^{-iPv(x+3\pi/2)}\psi(x + 3\pi/2, t)\bigg]. \tag{13}$$

These are the dominant resonances for weak potentials.

In Ref. [17], it was shown how the resonances for a kicked rotor at even multiples of π can be mapped onto a classical integrable dynamics. For a homogeneous initial distribution, no transport is expected in this case, but the inhomogeneous initial condition may selectively sample classical islands transporting in the positive direction. For the resonances at odd and half-integer multiples of π, the analysis of Ref. [17] cannot be applied.

Initial condition.—Figure 2 displays the time averaged wave number $\langle k \rangle_t$ of the wave packet over a number of temporal periods—which is proportional to the mean distance traveled—as a function of the effective Planck constant $\bar{k}$. Again we have chosen $\alpha = 0.3$. P is held constant, so that the potential strength in each of the panels of Fig. 2 varies along the abscissa as $K = P\bar{k}$.

Consider first the topmost panel, where the initial state is a zero-momentum homogeneous state. The curves are periodic in $\bar{k}$ with period 4π. To see why, note that when P is kept constant, only the kinetic part of the time evolution, $\exp(-i\bar{k}\hat{k}^2/2)$, depends on $\bar{k}$, and since k is restricted to integers, the change $\bar{k} \rightarrow \bar{k} + 4\pi$ does not affect the dynamics. Moreover, changing $\bar{k}$ to $-\bar{k}$ or $4\pi - \bar{k}$ results only in an overall complex conjugation of the wave function, hence the reflection symmetry about the point $\bar{k} = 2\pi$. For the dashed curve, the average is taken over the first 20 temporal periods. The mean drift is positive. This is, however, not a genuine ratchet effect, but rather a transient effect which is a consequence of the choice of initial state.

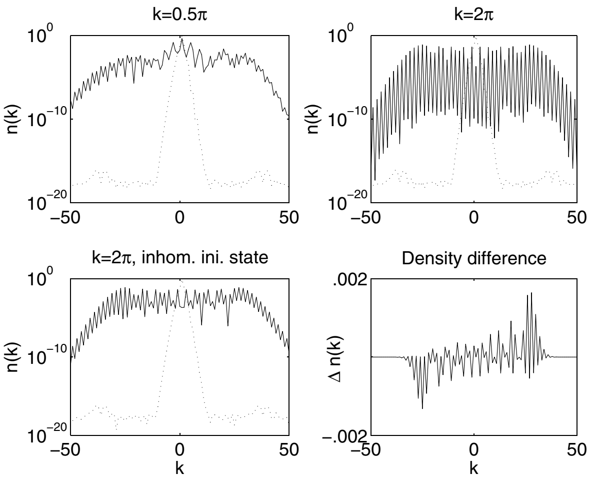

FIG. 3. Momentum distribution of the wave function. The top left panel displays the momentum distribution for the resonant case $\bar{k} = 0.5\pi$ after one (dotted line) and 100 kicks (solid line). The initial state was a homogeneous zero-momentum state. The top right panel shows the corresponding result for the case $\bar{k} = 2\pi$. The bottom left panel shows the corresponding result for $\bar{k} = 2\pi$ for inhomogeneous initial conditions, i.e., when the initial state is taken to be the ground state of the potential. The bottom right panel shows the difference between the momentum distributions for $\bar{k} = 2\pi$ with inhomogeneous and homogeneous initial conditions, $\Delta n = n_{\text{inhom}} - n_{\text{hom}}$.

As can be seen in Fig. 1, the momentum oscillates around zero when $\bar{k}$ is not at a resonant value. Indeed, when the averaging time is 100 kicks, the positive drift has averaged out to zero everywhere except for the resonances at $\bar{k}$ equal to half-integer multiples of π. The absence of drift at the resonances at integer multiples of π is in accordance with the analysis sketched above. However, whether a net drift is obtained depends also on the initial state. In the bottom panel, we display the result for the same system with a different initial condition, namely, the ground state of the asymmetric well. First, note that the figure lacks the symmetry seen in the upper panel, because the initial state depends nontrivially on $\bar{k}$. Next, we observe that in this case there are major resonances at all integer and half-integer multiples of π, and almost zero drift elsewhere. The drift is mostly positive: this is the direction expected for a classical diffusive ratchet in a sawtooth-shaped potential where the sawteeth lean to the left.

Figure 3 displays the momentum distributions for a few cases. The resonance at $\bar{k} = 0.5\pi$ displays a clearly asymmetric distribution after 100 kicks. The width of the momentum distribution is of the order $10\hbar k_L$, which for the parameters of Ref. [10] amounts to a velocity spread of 10 mm/s. For the resonance at $\bar{k} = 2\pi$, where no current is obtained, the momentum distribution is symmetric. Choosing the initial condition to be an eigenstate of the potential results, as we saw in Fig. 2, in transport but that is hard to observe in the momentum-distribution curve; in order to see this, one must subtract the corresponding momentum distribution for a homogeneous initial condition.

If experimentally an initial state is prepared to be a wave packet that extends over many optical-lattice sites and is smooth on the scale of one site, then the result of the top panel of Fig. 2 is to be expected, although the residual inhomogeneity will presumably result in smaller resonance peaks at the integer resonances. A mean momentum of the order $\hbar k_L$ over 100 kicks, as in Fig. 2, means for the parameter values used in Ref. [10], a mean velocity of about 1 mm/s, or a center-of-mass displacement of a few tens of lattice sites after 100 flashing periods.

In summary, we have investigated the conditions for directed transport for a quantum particle in a flashing ratchet potential. It is found that due to quantum resonances at specific values of the effective Planck constant, there can be net transport in a coherent, noise-free quantum system. The resonances occur when the effective Planck constant, which is proportional to the ratio between the recoil frequency and the frequency of the flashing, is equal to integer or half-integer multiples of π. The levels of precision, noise, and parameter control attainable in current experiments [10,15] should be sufficient to test these predictions in an optical-lattice setup.

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